

Decomposing graphs into 328 locally irregular graphs

Julien Bensmail^a, Martin Merker^b, Carsten Thomassen^c

a: Université Nice-Sophia-Antipolis, France

b: Universität Hamburg, Germany

c: Technical University of Denmark, Denmark

LIF, Marseille

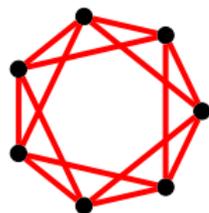
October 17th, 2016

General context

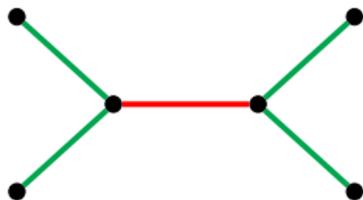
Definitions and main notions

G : undirected graph

G *locally irregular* = Every two adjacent vertices of G have distinct degrees



X



X

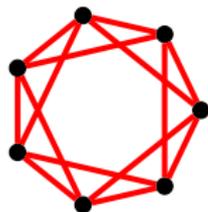


✓

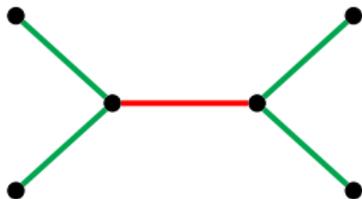
Definitions and main notions

G : undirected graph

G **locally irregular** = Every two adjacent vertices of G have distinct degrees



X



X



✓

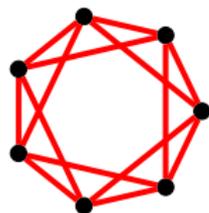
Decomposition of $G =$ Partition E_1, \dots, E_k of $E(G)$

Locally irregular decomposition = Decomposition into locally irregular graphs
(equivalently, **locally irregular edge-colouring**)

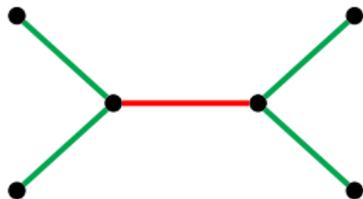
Definitions and main notions

G : undirected graph

G **locally irregular** = Every two adjacent vertices of G have distinct degrees



X



X



✓

Decomposition of $G =$ Partition E_1, \dots, E_k of $E(G)$

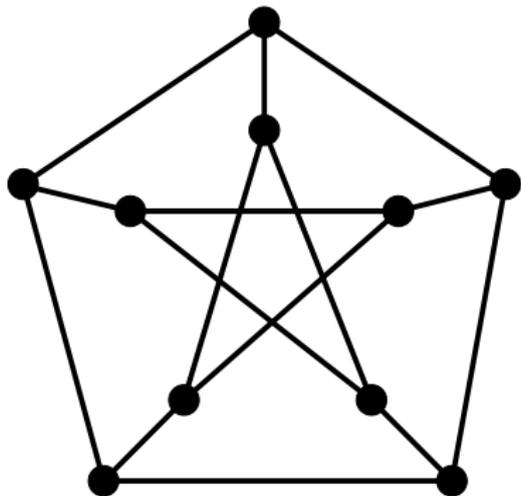
Locally irregular decomposition = Decomposition into locally irregular graphs
(equivalently, **locally irregular edge-colouring**)

$\chi'_{\text{irr}}(G)$ = Smallest $k \geq 1$, s.t. G has locally irregular k -edge-colourings

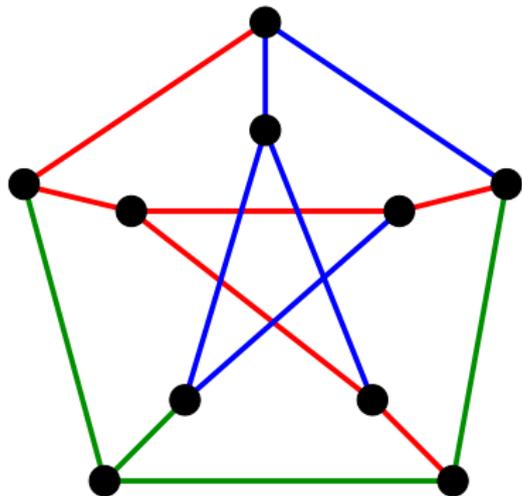
G **decomposable** = $\chi'_{\text{irr}}(G)$ exists.

G **exceptional**, otherwise.

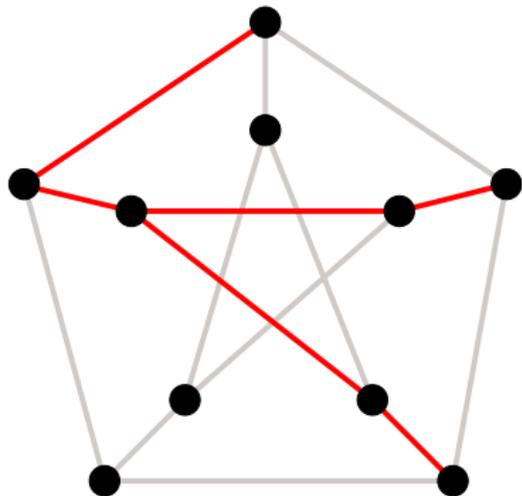
Example



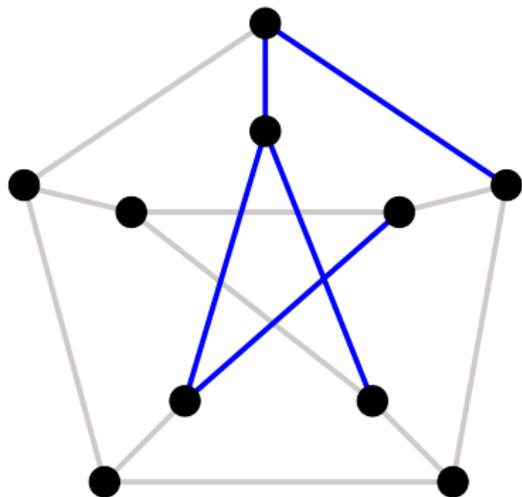
Example



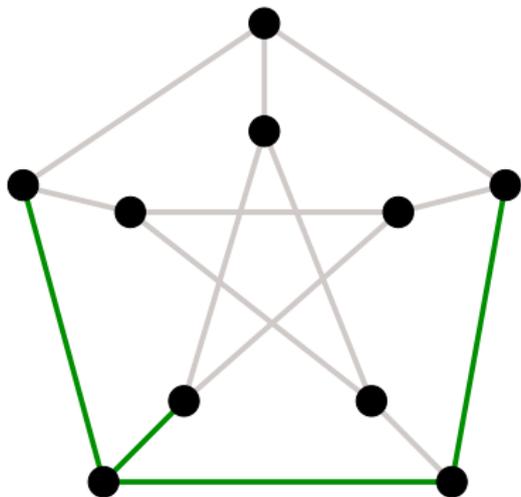
Example



Example



Example

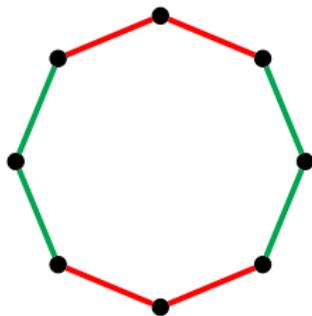


Some motivations

- 1 Local irregularity = Possible antonym notion to regularity
- 2 χ'_{irr} = Measure of closeness to irregularity

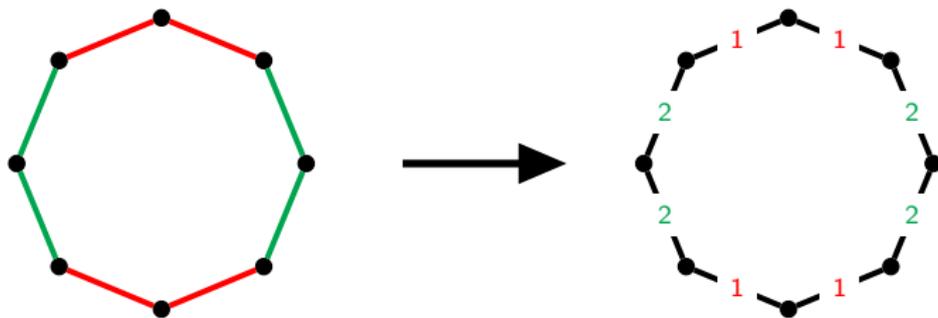
Some motivations

- 1 Local irregularity = Possible antonym notion to regularity
- 2 χ'_{irr} = Measure of closeness to irregularity
- 3 Connexions and applications to the [1-2-3 Conjecture](#)



Some motivations

- 1 Local irregularity = Possible antonym notion to regularity
- 2 χ'_{irr} = Measure of closeness to irregularity
- 3 Connexions and applications to the [1-2-3 Conjecture](#)



Previous works: Exceptional graphs

Exceptional graphs?

Previous works: Exceptional graphs

Exceptional graphs?

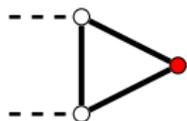
Some obvious ones: odd-length paths and odd-length cycles...

Previous works: Exceptional graphs

Exceptional graphs?

Some obvious ones: odd-length paths and odd-length cycles...

... but also \mathcal{T} :

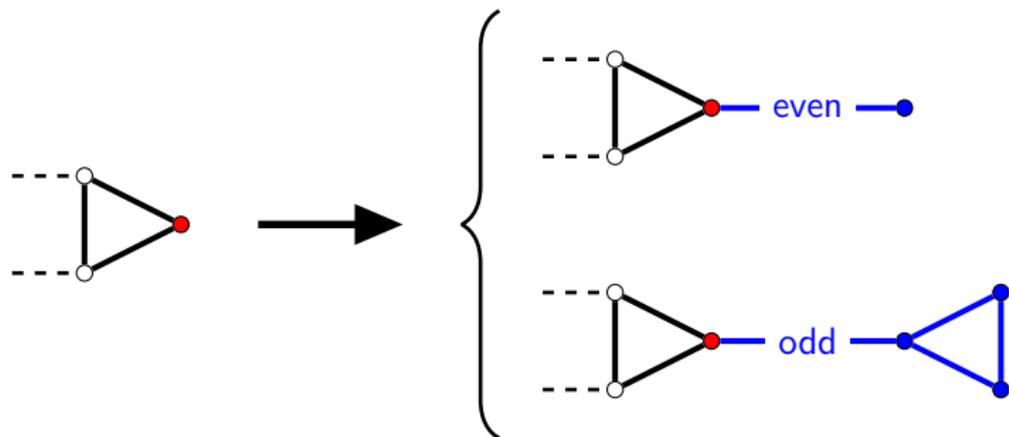


Previous works: Exceptional graphs

Exceptional graphs?

Some obvious ones: odd-length paths and odd-length cycles...

... but also \mathcal{T} :

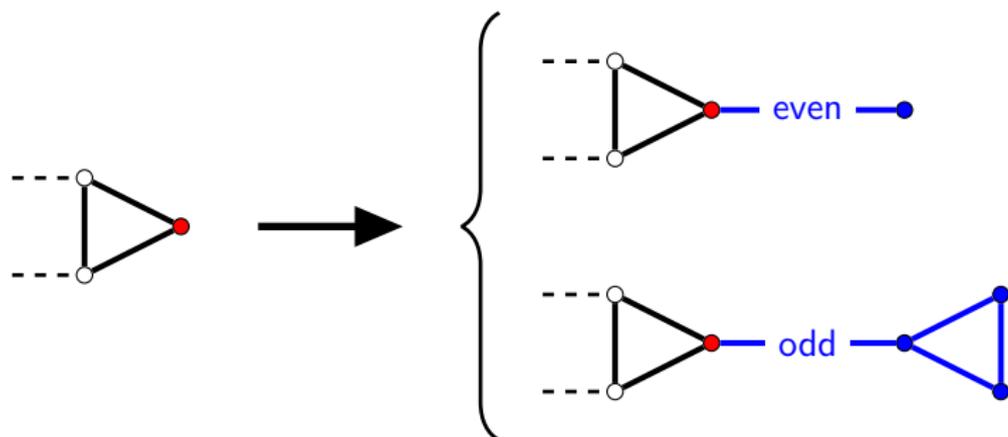


Previous works: Exceptional graphs

Exceptional graphs?

Some obvious ones: odd-length paths and odd-length cycles...

... but also \mathcal{T} :



Theorem – Baudon, B., Przybyło, Woźniak (2015)

Exceptional graphs are **exactly** these three classes of graphs.

Previous works: Main conjecture

How large can χ'_{irr} be?

Previous works: Main conjecture

How large can χ'_{irr} be?

Conjecture – Baudon, B., Przybyło, Woźniak (2015)

For every decomposable graph G , we have $\chi'_{\text{irr}}(G) \leq 3$.

Note: Would be tight (e.g. C_{4k+2} , K_n , etc.). Actually, unless $P = NP$, no “good” characterization of when $\chi'_{\text{irr}}(G) \leq 2$ [Baudon, B., Sopena (2015)].

Previous works: Main conjecture

How large can χ'_{irr} be?

Conjecture – Baudon, B., Przybyło, Woźniak (2015)

For every decomposable graph G , we have $\chi'_{\text{irr}}(G) \leq 3$.

Note: Would be tight (e.g. C_{4k+2} , K_n , etc.). Actually, unless $P = NP$, no “good” characterization of when $\chi'_{\text{irr}}(G) \leq 2$ [Baudon, B., Sopena (2015)].

Conjecture verified for:

- trees, regular bipartite graphs, $K_{n,m}$, K_n , some Cartesian products, regular graphs with degree $\geq 10^7$ [Baudon, B., Przybyło, Woźniak (2015)]

Previous works: Main conjecture

How large can χ'_{irr} be?

Conjecture – Baudon, B., Przybyło, Woźniak (2015)

For every decomposable graph G , we have $\chi'_{\text{irr}}(G) \leq 3$.

Note: Would be tight (e.g. C_{4k+2} , K_n , etc.). Actually, unless $P = NP$, no “good” characterization of when $\chi'_{\text{irr}}(G) \leq 2$ [Baudon, B., Sopena (2015)].

Conjecture verified for:

- trees, regular bipartite graphs, $K_{n,m}$, K_n , some Cartesian products, regular graphs with degree $\geq 10^7$ [Baudon, B., Przybyło, Woźniak (2015)]
- graphs with $\delta \geq 10^{10}$ [Przybyło (2016)]

Questions:

- 1 The conjecture for bipartite graphs?
- 2 General constant upper bounds on χ'_{irr} ?

Main questions, and partial answers

Questions:

- 1 The conjecture for bipartite graphs?
- 2 General constant upper bounds on χ'_{irr} ?

Today's (partial) answers:

Theorem – B., Merker, Thomassen (2016)

For every decomposable bipartite graph G , we have $\chi'_{\text{irr}}(G) \leq 10$.

For every decomposable graph G , we have $\chi'_{\text{irr}}(G) \leq 328$.

Theorem – B., Merker, Thomassen (2016)

For every decomposable bipartite graph G , we have $\chi'_{\text{irr}}(G) \leq 10$.

For every decomposable graph G , we have $\chi'_{\text{irr}}(G) \leq 328$.

General idea: Find edge-disjoint subgraphs G_1, \dots, G_k of G , s.t.

- $\chi'_{\text{irr}}(G - (E(G_1) \cup \dots \cup E(G_k)))$ is “small”
- $\chi'_{\text{irr}}(G_1), \dots, \chi'_{\text{irr}}(G_k)$ are “small”

⇒ Decompose the G_i 's and $G - (E(G_1) \cup \dots \cup E(G_k))$ independently

Main ideas and steps

Theorem – B., Merker, Thomassen (2016)

For every decomposable bipartite graph G , we have $\chi'_{\text{irr}}(G) \leq 10$.

For every decomposable graph G , we have $\chi'_{\text{irr}}(G) \leq 328$.

General idea: Find edge-disjoint subgraphs G_1, \dots, G_k of G , s.t.

- $\chi'_{\text{irr}}(G - (E(G_1) \cup \dots \cup E(G_k)))$ is “small”
- $\chi'_{\text{irr}}(G_1), \dots, \chi'_{\text{irr}}(G_k)$ are “small”

⇒ Decompose the G_i 's and $G - (E(G_1) \cup \dots \cup E(G_k))$ independently

Even-size graph = Graph whose all components have even size

Analogously, notion of **odd-size graph**

Main ideas and steps

Theorem – B., Merker, Thomassen (2016)

For every decomposable bipartite graph G , we have $\chi'_{\text{irr}}(G) \leq 10$.

For every decomposable graph G , we have $\chi'_{\text{irr}}(G) \leq 328$.

General idea: Find edge-disjoint subgraphs G_1, \dots, G_k of G , s.t.

- $\chi'_{\text{irr}}(G - (E(G_1) \cup \dots \cup E(G_k)))$ is “small”
- $\chi'_{\text{irr}}(G_1), \dots, \chi'_{\text{irr}}(G_k)$ are “small”

⇒ Decompose the G_i 's and $G - (E(G_1) \cup \dots \cup E(G_k))$ independently

Even-size graph = Graph whose all components have even size

Analogously, notion of **odd-size graph**

Main steps:

- 1 Reducing the conjecture to even-size graphs

Main ideas and steps

Theorem – B., Merker, Thomassen (2016)

For every decomposable bipartite graph G , we have $\chi'_{\text{irr}}(G) \leq 10$.

For every decomposable graph G , we have $\chi'_{\text{irr}}(G) \leq 328$.

General idea: Find edge-disjoint subgraphs G_1, \dots, G_k of G , s.t.

- $\chi'_{\text{irr}}(G - (E(G_1) \cup \dots \cup E(G_k)))$ is “small”
- $\chi'_{\text{irr}}(G_1), \dots, \chi'_{\text{irr}}(G_k)$ are “small”

⇒ Decompose the G_i 's and $G - (E(G_1) \cup \dots \cup E(G_k))$ independently

Even-size graph = Graph whose all components have even size

Analogously, notion of **odd-size graph**

Main steps:

- 1 Reducing the conjecture to even-size graphs
- 2 Decomposing even-size bipartite graphs

Main ideas and steps

Theorem – B., Merker, Thomassen (2016)

For every decomposable bipartite graph G , we have $\chi'_{\text{irr}}(G) \leq 10$.

For every decomposable graph G , we have $\chi'_{\text{irr}}(G) \leq 328$.

General idea: Find edge-disjoint subgraphs G_1, \dots, G_k of G , s.t.

- $\chi'_{\text{irr}}(G - (E(G_1) \cup \dots \cup E(G_k)))$ is “small”
- $\chi'_{\text{irr}}(G_1), \dots, \chi'_{\text{irr}}(G_k)$ are “small”

⇒ Decompose the G_i 's and $G - (E(G_1) \cup \dots \cup E(G_k))$ independently

Even-size graph = Graph whose all components have even size

Analogously, notion of **odd-size graph**

Main steps:

- 1 Reducing the conjecture to even-size graphs
- 2 Decomposing even-size bipartite graphs
- 3 Using Przybyło's and the bipartite results

Step 1: Reducing to even-size graphs

Reducing to even-size graphs

Point: Avoids dealing with exceptional graphs

Reducing to even-size graphs

Point: Avoids dealing with exceptional graphs

Meaning: There exists $k \geq 1$ small, s.t.

$$\chi'_{\text{irr}}(\text{odd-size, decomposable}) \leq \chi'_{\text{irr}}(\text{even-size}) + k$$

Reducing to even-size graphs

Point: Avoids dealing with exceptional graphs

Meaning: There exists $k \geq 1$ small, s.t.

$$\chi'_{\text{irr}}(\text{odd-size, decomposable}) \leq \chi'_{\text{irr}}(\text{even-size}) + k$$

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Reducing to even-size graphs

Point: Avoids dealing with exceptional graphs

Meaning: There exists $k \geq 1$ small, s.t.

$$\chi'_{\text{irr}}(\text{odd-size, decomposable}) \leq \chi'_{\text{irr}}(\text{even-size}) + k$$

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Corollary

$$\chi'_{\text{irr}}(\text{odd-size, decomposable}) \leq \chi'_{\text{irr}}(\text{even-size}) + 1$$

Two useful easy lemmas

First lemma

Let G be a connected odd-size graph. For every vertex $v \in V(G)$, there exists an edge e incident to v , s.t. $G - e$ is an even-size graph.

Proof. Assume all edges incident to v are cut-edges. Then consider the sizes of the components incident to v . ■

Two useful easy lemmas

First lemma

Let G be a connected odd-size graph. For every vertex $v \in V(G)$, there exists an edge e incident to v , s.t. $G - e$ is an even-size graph.

Proof. Assume all edges incident to v are cut-edges. Then consider the sizes of the components incident to v . ■

Second lemma

Let G be a connected even-size graph. For every vertex $v \in V(G)$, there exists a path P of length 2 containing v , s.t. $G - E(P)$ is an even-size graph.

Proof. Consider any edge uv incident to v , and apply the previous lemma to an odd-size component incident to u or v in $G - uv$. ■

Finding a or in an odd-size decomposable graph

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample.

Finding a or in an odd-size decomposable graph

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample. Then:

- $\Delta(G) \geq 3$;

Finding a or in an odd-size decomposable graph

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample. Then:

- $\Delta(G) \geq 3$;
- Every 3^+ -vertex is a cut-vertex;

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample. Then:

- $\Delta(G) \geq 3$;
- Every 3^+ -vertex is a cut-vertex;
- No cycle with length at least 4;

Finding a or in an odd-size decomposable graph

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample. Then:

- $\Delta(G) \geq 3$;
- Every 3^+ -vertex is a cut-vertex;
- No cycle with length at least 4;
- No intersecting triangles;

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample. Then:

- $\Delta(G) \geq 3$;
- Every 3^+ -vertex is a cut-vertex;
- No cycle with length at least 4;
- No intersecting triangles;
- No induced claw $\Rightarrow \Delta(G) \leq 3$ and every 3-vertex lies in a triangle;

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample. Then:

- $\Delta(G) \geq 3$;
- Every 3^+ -vertex is a cut-vertex;
- No cycle with length at least 4;
- No intersecting triangles;
- No induced claw $\Rightarrow \Delta(G) \leq 3$ and every 3-vertex lies in a triangle;
- Pendant paths have even length;

Finding a or in an odd-size decomposable graph

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample. Then:

- $\Delta(G) \geq 3$;
- Every 3^+ -vertex is a cut-vertex;
- No cycle with length at least 4;
- No intersecting triangles;
- No induced claw $\Rightarrow \Delta(G) \leq 3$ and every 3-vertex lies in a triangle;
- Pendant paths have even length;
- Triangles are joined by odd length paths.

Finding a or in an odd-size decomposable graph

Theorem

In every odd-size decomposable graph, one can find a  or a  whose deletion leaves an even-size graph.

Proof. Assume G is a counterexample. Then:

- $\Delta(G) \geq 3$;
- Every 3^+ -vertex is a cut-vertex;
- No cycle with length at least 4;
- No intersecting triangles;
- No induced claw $\Rightarrow \Delta(G) \leq 3$ and every 3-vertex lies in a triangle;
- Pendant paths have even length;
- Triangles are joined by odd length paths.

$\Rightarrow G$ is exceptional, a contradiction. ■

Step 2: Decomposing even-size bipartite graphs

Theorem

$\chi'_{\text{irr}}(\text{even-size, bipartite}) \leq 9$. Consequently,

$$\chi'_{\text{irr}}(\text{decomposable, bipartite}) \leq 10.$$

Main result, and proof ideas

Theorem

$\chi'_{\text{irr}}(\text{even-size, bipartite}) \leq 9$. Consequently,

$$\chi'_{\text{irr}}(\text{decomposable, bipartite}) \leq 10.$$

$G = (A, B)$: even-size bipartite graph

Main result, and proof ideas

Theorem

$\chi'_{\text{irr}}(\text{even-size, bipartite}) \leq 9$. Consequently,

$$\chi'_{\text{irr}}(\text{decomposable, bipartite}) \leq 10.$$

$G = (A, B)$: even-size bipartite graph

Note: Degrees in A even + Degrees in B odd $\Rightarrow G$ locally irregular!

Main result, and proof ideas

Theorem

$\chi'_{\text{irr}}(\text{even-size, bipartite}) \leq 9$. Consequently,

$$\chi'_{\text{irr}}(\text{decomposable, bipartite}) \leq 10.$$

$G = (A, B)$: even-size bipartite graph

Note: Degrees in A even + Degrees in B odd $\Rightarrow G$ locally irregular!

Main idea: Make G as close as possible to this structure, i.e. remove two decomposable subgraphs F_A and F_B with small χ'_{irr} , s.t.

- 1 all degrees in A get even
- 2 all degrees in B get odd

Main result, and proof ideas

Theorem

$\chi'_{\text{irr}}(\text{even-size, bipartite}) \leq 9$. Consequently,

$$\chi'_{\text{irr}}(\text{decomposable, bipartite}) \leq 10.$$

$G = (A, B)$: even-size bipartite graph

Note: Degrees in A even + Degrees in B odd $\Rightarrow G$ locally irregular!

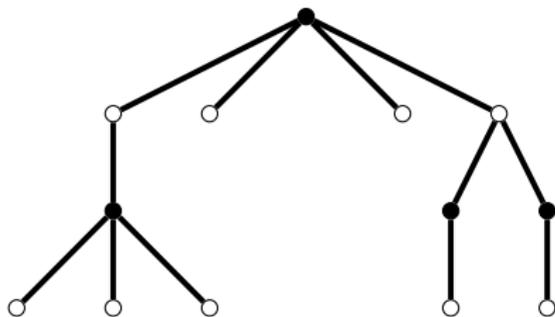
Main idea: Make G as close as possible to this structure, i.e. remove two decomposable subgraphs F_A and F_B with small χ'_{irr} , s.t.

- 1 all degrees in A get even
- 2 all degrees in B get odd

\Rightarrow We do that with F_A and F_B being two **balanced forests**

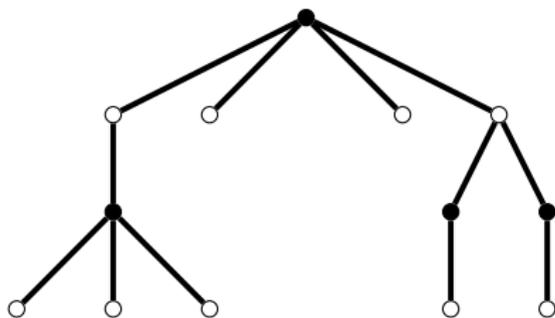
Balanced forests

Balanced forest = In one of the two colour classes, only even degrees



Balanced forests

Balanced forest = In one of the two colour classes, only even degrees

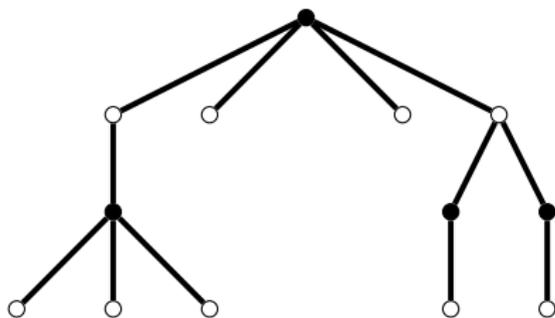


Change the degree parities of pairs of vertices of G in the same part?

⇒ Find a balanced forest!

Balanced forests

Balanced forest = In one of the two colour classes, only even degrees



Change the degree parities of pairs of vertices of G in the same part?

⇒ Find a balanced forest!

Lemma

$$\chi'_{\text{irr}}(\text{balanced tree}) \leq 2$$

(though infinitely many trees verify $\chi'_{\text{irr}} = 3$ [Baudon, B., Sopena (2015)])

Getting “almost” the desired degree properties

Reminder: G is an even-size bipartite graph

Getting “almost” the desired degree properties

Reminder: G is an even-size bipartite graph

Theorem

There exists a balanced forest F_A with leaves in A , s.t., in $G - E(F_A)$, all vertices in A have even degree.

Proof. In a spanning forest, take a system F_A of edge-disjoint paths joining pairs of odd-degree vertices in A , which minimizes the total length. ■

Getting “almost” the desired degree properties

Reminder: G is an even-size bipartite graph

Theorem

There exists a **balanced forest** F_A with leaves in A , s.t., in $G - E(F_A)$, all vertices in A have even degree.

Proof. In a spanning forest, take a system F_A of edge-disjoint paths joining pairs of odd-degree vertices in A , which minimizes the total length. ■

Theorem

There exists a **balanced forest** F_B with leaves in B , s.t., in $G - E(F_B)$, all vertices of B , but at most one, have odd degree.

Proof. Just do the same, but with a system F_B of edge-disjoint paths joining pairs of even-degree vertices in B . ■

It cannot be that easy...

Note: Degrees in A even $\Rightarrow G$ remains decomposable

It cannot be that easy...

Note: Degrees in A even $\Rightarrow G$ remains decomposable

A remaining even-degree vertex v in B ?

It cannot be that easy...

Note: Degrees in A even $\Rightarrow G$ remains decomposable

A remaining even-degree vertex v in B ?

\Rightarrow Remove “something” with small χ'_{irr} , so that we get the desired structure

It cannot be that easy...

Note: Degrees in A even $\Rightarrow G$ remains decomposable

A remaining even-degree vertex v in B ?

\Rightarrow Remove “something” with small χ'_{irr} , so that we get the desired structure

Case 1. No cycle through v .

\Rightarrow All edges incident to v are cut-edges. Choose any e of them, and let H_1 and H_2 be the two components of $G - e$. Note that $d_{H_1+e}(v)$ and $d_{H_2}(v)$ are odd. Hence $H_1 + e$ and H_2 are locally irregular, and

$$\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(H_1 + e) + \chi'_{\text{irr}}(H_2) + \chi'_{\text{irr}}(F_A) + \chi'_{\text{irr}}(F_B) \leq 6.$$

It cannot be that easy...

Note: Degrees in A even $\Rightarrow G$ remains decomposable

A remaining even-degree vertex v in B ?

\Rightarrow Remove “something” with small χ'_{irr} , so that we get the desired structure

Case 1. No cycle through v .

\Rightarrow All edges incident to v are cut-edges. Choose any e of them, and let H_1 and H_2 be the two components of $G - e$. Note that $d_{H_1+e}(v)$ and $d_{H_2}(v)$ are odd. Hence $H_1 + e$ and H_2 are locally irregular, and

$$\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(H_1 + e) + \chi'_{\text{irr}}(H_2) + \chi'_{\text{irr}}(F_A) + \chi'_{\text{irr}}(F_B) \leq 6.$$

Case 2. A cycle C through v .

\Rightarrow Remove C from G ; this does not alter the degrees' parities.

Lemma

There exists a path P starting at v , s.t. $G - E(P)$ is locally irregular.

Proof. Focus on the connected component containing v . Set $P = \emptyset$ and $v_0 = v$. If $G - E(P)$ is not locally irregular, then there is a $v_1 \in A$ such that $d(v_0) = d(v_1)$. Then move v_0v_1 to P . If $G - E(P)$ is not locally irregular, then there is a $v_2 \in B$ such that $d(v_1) = d(v_2)$. Then move v_1v_2 to P . Repeat this process until it stops.

Lemma

There exists a path P starting at v , s.t. $G - E(P)$ is locally irregular.

Proof. Focus on the connected component containing v . Set $P = \emptyset$ and $v_0 = v$. If $G - E(P)$ is not locally irregular, then there is a $v_1 \in A$ such that $d(v_0) = d(v_1)$. Then move v_0v_1 to P . If $G - E(P)$ is not locally irregular, then there is a $v_2 \in B$ such that $d(v_1) = d(v_2)$. Then move v_1v_2 to P . Repeat this process until it stops. Note that the conflicting degrees strictly decrease. Hence all v_i 's are different, and the process ends at some point. Furthermore, P induces a path. ■

Lemma

There exists a path P starting at v , s.t. $G - E(P)$ is locally irregular.

Proof. Focus on the connected component containing v . Set $P = \emptyset$ and $v_0 = v$. If $G - E(P)$ is not locally irregular, then there is a $v_1 \in A$ such that $d(v_0) = d(v_1)$. Then move v_0v_1 to P . If $G - E(P)$ is not locally irregular, then there is a $v_2 \in B$ such that $d(v_1) = d(v_2)$. Then move v_1v_2 to P . Repeat this process until it stops. Note that the conflicting degrees strictly decrease. Hence all v_i 's are different, and the process ends at some point. Furthermore, P induces a path. ■

P exceptional?

Lemma

There exists a path P starting at v , s.t. $G - E(P)$ is locally irregular.

Proof. Focus on the connected component containing v . Set $P = \emptyset$ and $v_0 = v$. If $G - E(P)$ is not locally irregular, then there is a $v_1 \in A$ such that $d(v_0) = d(v_1)$. Then move v_0v_1 to P . If $G - E(P)$ is not locally irregular, then there is a $v_2 \in B$ such that $d(v_1) = d(v_2)$. Then move v_1v_2 to P . Repeat this process until it stops.

Note that the conflicting degrees strictly decrease. Hence all v_i 's are different, and the process ends at some point. Furthermore, P induces a path. ■

P exceptional? Well, yes, but...

Observation

$$\chi'_{\text{irr}}(C \cup P) \leq 4$$

Gathering everything

Call G' what remains of G ;

Gathering everything

Call G' what remains of G ; then

$$\chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(G') + \chi'_{\text{irr}}(F_A) + \chi'_{\text{irr}}(F_B) + \chi'_{\text{irr}}(C \cup P) \leq 9$$



Step 3: Using Przybyło's and the bipartite results

Main result, and rough proof ideas

Theorem

$$\chi'_{\text{irr}}(\text{decomposable}) \leq 328$$

Main result, and rough proof ideas

Theorem

$$\chi'_{\text{irr}}(\text{decomposable}) \leq 328$$

G : decomposable **even-size** graph (thus, need to show 327 only)
We use both the result on decomposable bipartite graphs, and

Theorem – Przybyło (2016)

Let H be a graph with $\delta(H) \geq 10^{10}$. Then $\chi'_{\text{irr}}(H) \leq 3$.

Main result, and rough proof ideas

Theorem

$$\chi'_{\text{irr}}(\text{decomposable}) \leq 328$$

G : decomposable **even-size** graph (thus, need to show 327 only)
We use both the result on decomposable bipartite graphs, and

Theorem – Przybyło (2016)

Let H be a graph with $\delta(H) \geq 10^{10}$. Then $\chi'_{\text{irr}}(H) \leq 3$.

(Rough) ideas:

- 1 Decompose G into $H + D$, where:
 - $\delta(H) \geq 10^{10}$
 - D is an **even-size** $(2 \cdot 10^{10} + 2)$ -degenerate graph

Main result, and rough proof ideas

Theorem

$$\chi'_{\text{irr}}(\text{decomposable}) \leq 328$$

G : decomposable **even-size** graph (thus, need to show 327 only)
We use both the result on decomposable bipartite graphs, and

Theorem – Przybyło (2016)

Let H be a graph with $\delta(H) \geq 10^{10}$. Then $\chi'_{\text{irr}}(H) \leq 3$.

(Rough) ideas:

- 1 Decompose G into $H + D$, where:
 - $\delta(H) \geq 10^{10}$
 - D is an **even-size** $(2 \cdot 10^{10} + 2)$ -degenerate graph
- 2 Decompose D into $\log_2(2 \cdot 10^{10} + 3) + 1$ even-size bipartite graphs

Main result, and rough proof ideas

Theorem

$$\chi'_{\text{irr}}(\text{decomposable}) \leq 328$$

G : decomposable **even-size** graph (thus, need to show 327 only)
We use both the result on decomposable bipartite graphs, and

Theorem – Przybyło (2016)

Let H be a graph with $\delta(H) \geq 10^{10}$. Then $\chi'_{\text{irr}}(H) \leq 3$.

(Rough) ideas:

- 1 Decompose G into $H + D$, where:
 - $\delta(H) \geq 10^{10}$
 - D is an even-size $(2 \cdot 10^{10} + 2)$ -degenerate graph
 - 2 Decompose D into $\log_2(2 \cdot 10^{10} + 3) + 1$ even-size bipartite graphs
- $\Rightarrow \chi'_{\text{irr}}(G) \leq \chi'_{\text{irr}}(H) + \chi'_{\text{irr}}(D) \leq 3 + 9 \cdot 36 = 327$

Step 1: Getting H and D

We want:

- $\delta(H) \geq 10^{10}$
- D is an even-size $(2 \cdot 10^{10} + 2)$ -degenerate graph

Step 1: Getting H and D

We want:

- $\delta(H) \geq 10^{10}$
- D is an even-size $(2 \cdot 10^{10} + 2)$ -degenerate graph

Start from $H = G$, and D being the empty graph. Repeat the following: Until H verifies $\delta(H) > 2 \cdot 10^{10} + 2$, move to D a vertex (and its incident edges) with degree at most $2 \cdot 10^{10} + 2$ in H .

Step 1: Getting H and D

We want:

- $\delta(H) \geq 10^{10}$
- D is an even-size $(2 \cdot 10^{10} + 2)$ -degenerate graph

Start from $H = G$, and D being the empty graph. Repeat the following: Until H verifies $\delta(H) > 2 \cdot 10^{10} + 2$, move to D a vertex (and its incident edges) with degree at most $2 \cdot 10^{10} + 2$ in H . Once finished, we have:

- $\delta(H) > 2 \cdot 10^{10} + 2$,
- and D is $(2 \cdot 10^{10} + 2)$ -degenerate.

Step 1: Getting H and D

We want:

- $\delta(H) \geq 10^{10}$
- D is an even-size $(2 \cdot 10^{10} + 2)$ -degenerate graph

Start from $H = G$, and D being the empty graph. Repeat the following: Until H verifies $\delta(H) > 2 \cdot 10^{10} + 2$, move to D a vertex (and its incident edges) with degree at most $2 \cdot 10^{10} + 2$ in H . Once finished, we have:

- $\delta(H) > 2 \cdot 10^{10} + 2$,
- and D is $(2 \cdot 10^{10} + 2)$ -degenerate.

Issue: D might have odd-size components!

Step 1: Getting H and D

We want:

- $\delta(H) \geq 10^{10}$
- D is an even-size $(2 \cdot 10^{10} + 2)$ -degenerate graph

Start from $H = G$, and D being the empty graph. Repeat the following: Until H verifies $\delta(H) > 2 \cdot 10^{10} + 2$, move to D a vertex (and its incident edges) with degree at most $2 \cdot 10^{10} + 2$ in H . Once finished, we have:

- $\delta(H) > 2 \cdot 10^{10} + 2$,
- and D is $(2 \cdot 10^{10} + 2)$ -degenerate.

Issue: D might have odd-size components!

Solution: For every component of D , “steal” an incident edge from H

Step 1: Getting H and D

We want:

- $\delta(H) \geq 10^{10}$
- D is an even-size $(2 \cdot 10^{10} + 2)$ -degenerate graph

Start from $H = G$, and D being the empty graph. Repeat the following: Until H verifies $\delta(H) > 2 \cdot 10^{10} + 2$, move to D a vertex (and its incident edges) with degree at most $2 \cdot 10^{10} + 2$ in H . Once finished, we have:

- $\delta(H) > 2 \cdot 10^{10} + 2$,
- and D is $(2 \cdot 10^{10} + 2)$ -degenerate.

Issue: D might have odd-size components!

Solution: For every component of D , “steal” an incident edge from H

Issue: A vertex from H might lose a lot of its degree

Step 1: Getting H and D

We want:

- $\delta(H) \geq 10^{10}$
- D is an even-size $(2 \cdot 10^{10} + 2)$ -degenerate graph

Start from $H = G$, and D being the empty graph. Repeat the following: Until H verifies $\delta(H) > 2 \cdot 10^{10} + 2$, move to D a vertex (and its incident edges) with degree at most $2 \cdot 10^{10} + 2$ in H . Once finished, we have:

- $\delta(H) > 2 \cdot 10^{10} + 2$,
- and D is $(2 \cdot 10^{10} + 2)$ -degenerate.

Issue: D might have odd-size components!

Solution: For every component of D , “steal” an incident edge from H

Issue: A vertex from H might loose a lot of its degree

Solution: Orient the edges of H in a balanced way, and “steal” out-edges only

Step 1: Getting H and D

We want:

- $\delta(H) \geq 10^{10}$
- D is an even-size $(2 \cdot 10^{10} + 2)$ -degenerate graph

Start from $H = G$, and D being the empty graph. Repeat the following: Until H verifies $\delta(H) > 2 \cdot 10^{10} + 2$, move to D a vertex (and its incident edges) with degree at most $2 \cdot 10^{10} + 2$ in H . Once finished, we have:

- $\delta(H) > 2 \cdot 10^{10} + 2$,
- and D is $(2 \cdot 10^{10} + 2)$ -degenerate.

Issue: D might have odd-size components!

Solution: For every component of D , “steal” an incident edge from H

Issue: A vertex from H might loose a lot of its degree

Solution: Orient the edges of H in a balanced way, and “steal” out-edges only

Note: Eventually, $\delta(H) \geq 10^{10}$ and D remains $(2 \cdot 10^{10} + 2)$ -degenerate

Step 2: Decomposing D into even-size bipartite graphs

Lemma

Let v be a vertex with degree d even in a given graph G' . If $G' - v$ decomposes into $\lceil \log_2 d \rceil + 1$ even-size bipartite graphs, then so does G' .

Proof. By induction on d . If $d = 2$, then we may assume, in a decomposition of $G' - v$, that the two neighbours u_1 and u_2 are joined by odd-length paths in the bipartite 1- and 2-subgraphs.

Step 2: Decomposing D into even-size bipartite graphs

Lemma

Let v be a vertex with degree d even in a given graph G' . If $G' - v$ decomposes into $\lceil \log_2 d \rceil + 1$ even-size bipartite graphs, then so does G' .

Proof. By induction on d . If $d = 2$, then we may assume, in a decomposition of $G' - v$, that the two neighbours u_1 and u_2 are joined by odd-lengths paths in the bipartite 1- and 2-subgraphs.

Note further that, along one such path in the 2-subgraph, it cannot be that every two subsequent vertices are joined by an even-length path in the 1-subgraph. Choose the first pair which does not verify this, and change the colour of the joining edge to 1. Each of the 1- and 2-subgraphs now has one odd-size component, but we can add to them the convenient edge incident to v .

Step 2: Decomposing D into even-size bipartite graphs

Lemma

Let v be a vertex with degree d even in a given graph G' . If $G' - v$ decomposes into $\lceil \log_2 d \rceil + 1$ even-size bipartite graphs, then so does G' .

Proof. By induction on d . If $d = 2$, then we may assume, in a decomposition of $G' - v$, that the two neighbours u_1 and u_2 are joined by odd-length paths in the bipartite 1- and 2-subgraphs.

Note further that, along one such path in the 2-subgraph, it cannot be that every two subsequent vertices are joined by an even-length path in the 1-subgraph. Choose the first pair which does not verify this, and change the colour of the joining edge to 1. Each of the 1- and 2-subgraphs now has one odd-size component, but we can add to them the convenient edge incident to v .

For the induction step, we consider $G' - v$, and repeatedly consider one of the $\lceil \log_2 d \rceil + 1$ even-size bipartite graphs, and add as many edges to them. This number is about $\frac{d(v)}{2}$, $\frac{d(v)}{4}$, and so on. ■

Step 2: Decomposing D into even-size bipartite graphs

Lemma

Let G' be a even-size d -degenerate graph. Then G' can be decomposed into $\lceil \log_2(d + 1) \rceil + 1$ even-size bipartite graphs.

Proof. Let G' be a smallest counterexample. Note that when removing a cut-vertex, two components result, one of which consists of one vertex.

Step 2: Decomposing D into even-size bipartite graphs

Lemma

Let G' be a even-size d -degenerate graph. Then G' can be decomposed into $\lceil \log_2(d+1) \rceil + 1$ even-size bipartite graphs.

Proof. Let G' be a smallest counterexample. Note that when removing a cut-vertex, two components result, one of which consists of one vertex.

Let v be a vertex with smallest degree different from 1. If $d(v)$ is even, just apply the previous lemma. Consider thus the case where $d(v)$ is odd.

Step 2: Decomposing D into even-size bipartite graphs

Lemma

Let G' be a *even-size d -degenerate graph*. Then G' can be decomposed into $\lceil \log_2(d+1) \rceil + 1$ *even-size bipartite graphs*.

Proof. Let G' be a smallest counterexample. Note that when removing a cut-vertex, two components result, one of which consists of one vertex.

Let v be a vertex with smallest degree different from 1. If $d(v)$ is even, just apply the previous lemma. Consider thus the case where $d(v)$ is odd.

We consider $G' - v$. If there is no isolated vertex w , we add such a vertex w , and an edge joining w and another neighbour u of v . So, $G' - v + uw$ has even size, is d -degenerate, and hence admits a decomposition into $\lceil \log_2(d+1) \rceil + 1$ even-size bipartite graphs. In G that decomposition is good, except that there is an odd-size bipartite graph containing u . Then just add a “large” odd number of edges incident to v to that odd-size bipartite graph. It remains an even number of edges incident to v , that we can add to the other even-size bipartite graphs. ■

Conclusion and perspectives

- ① Improving our proof scheme?
 - Not using Przybyło's result?
 - Using other kinds of auxiliary decompositions?

Conclusion and perspectives

1 Improving our proof scheme?

- Not using Przybyło's result?
- Using other kinds of auxiliary decompositions?

2 Bipartite graphs?

- $\chi'_{\text{irr}}(\text{path}+\text{cycle}) \leq 3$?
- Other method not based on even-odd degrees?

- 1 Improving our proof scheme?
 - Not using Przybyło's result?
 - Using other kinds of auxiliary decompositions?
- 2 Bipartite graphs?
 - $\chi'_{\text{irr}}(\text{path}+\text{cycle}) \leq 3$?
 - Other method not based on even-odd degrees?
- 3 Other classes of graphs?
 - Bounded degree?
 - Sparse classes?

Conclusion and perspectives

1 Improving our proof scheme?

- Not using Przybyło's result?
- Using other kinds of auxiliary decompositions?

2 Bipartite graphs?

- $\chi'_{\text{irr}}(\text{path}+\text{cycle}) \leq 3$?
- Other method not based on even-odd degrees?

3 Other classes of graphs?

- Bounded degree?
- Sparse classes?

4 When does $\chi'_{\text{irr}}(G) \leq 2$ hold?

- Bipartite graphs?
- Large degree?

Conclusion and perspectives

- 1 Improving our proof scheme?
 - Not using Przybyło's result?
 - Using other kinds of auxiliary decompositions?
- 2 Bipartite graphs?
 - $\chi'_{\text{irr}}(\text{path}+\text{cycle}) \leq 3$?
 - Other method not based on even-odd degrees?
- 3 Other classes of graphs?
 - Bounded degree?
 - Sparse classes?
- 4 When does $\chi'_{\text{irr}}(G) \leq 2$ hold?
 - Bipartite graphs?
 - Large degree?

Thank you for your attention!