

Decomposing graphs into locally irregular subgraphs: allowing K_2 's helps a lot

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Motivation: the 1-2-3 Conjecture

G : undirected simple graph

w : edge-weighting of G

Definitions: *weighted degree, neighbour-sum-distinguishing weighting*

For every vertex $v \in V(G)$, the *weighted degree* s_w of v (by w) is

$$s_w(v) := \sum_{u \in N(v)} w(vu).$$

w is *neighbour-sum-distinguishing* if s_w is proper.

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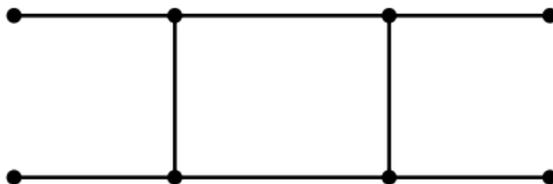
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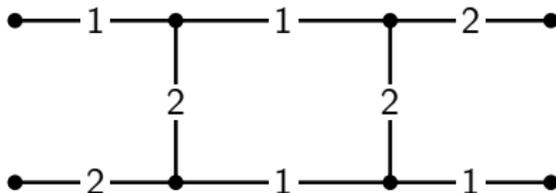
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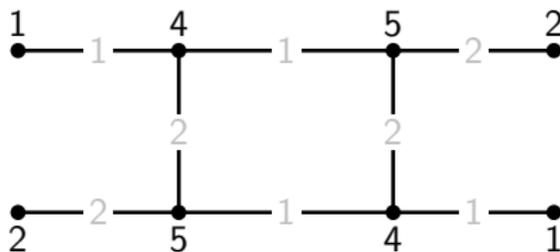
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No neighbour-sum-distinguishing edge-weighting of K_2

1-2-3 Conjecture [Karoński, Łuczak, Thomason – 2004]

Every graph with no component isomorphic to K_2 admits a neighbour-sum-distinguishing 3-edge-weighting.

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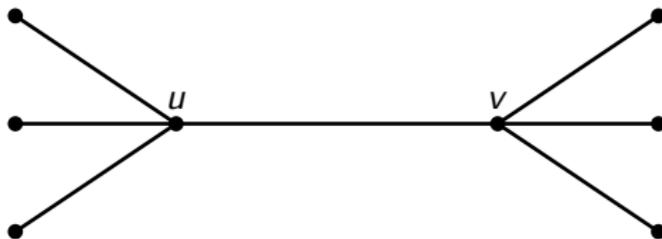
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- ... but $\{1, 2, 3\}$ may be needed (for e.g. C_6)
- $\{1, 2, 3, 4, 5\}$ is always sufficient [Kalkowski, Karoński, Pfender – 2010]

What about regular graphs?

Intuitively harder: regularity has to be “broken”

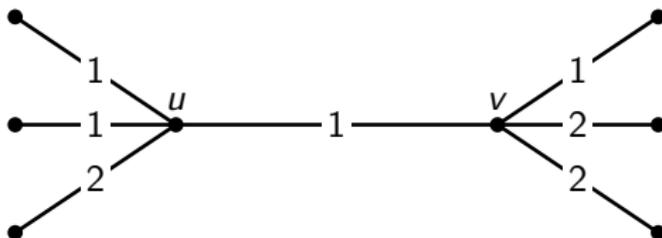
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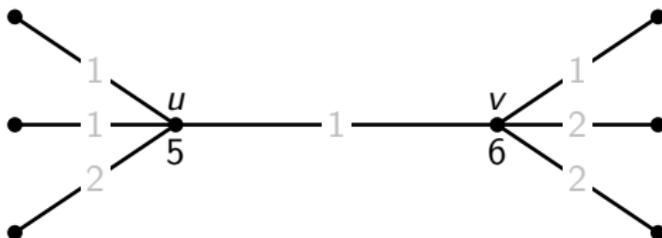
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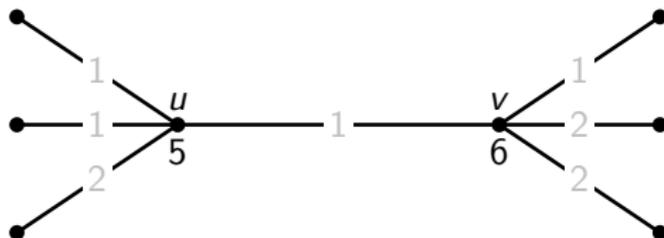
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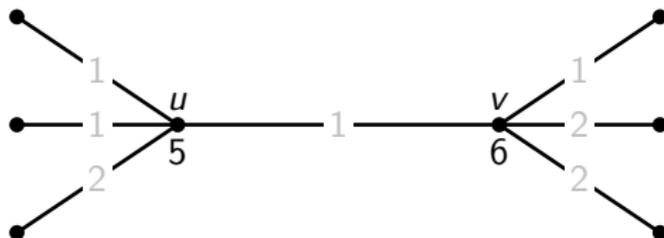


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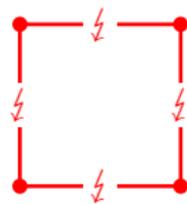
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\Rightarrow edge-colouring inducing *locally irregular subgraphs*

Locally irregular edge-colouring

Definition: *locally irregular graph*

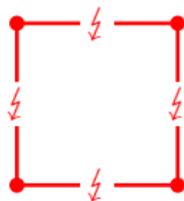
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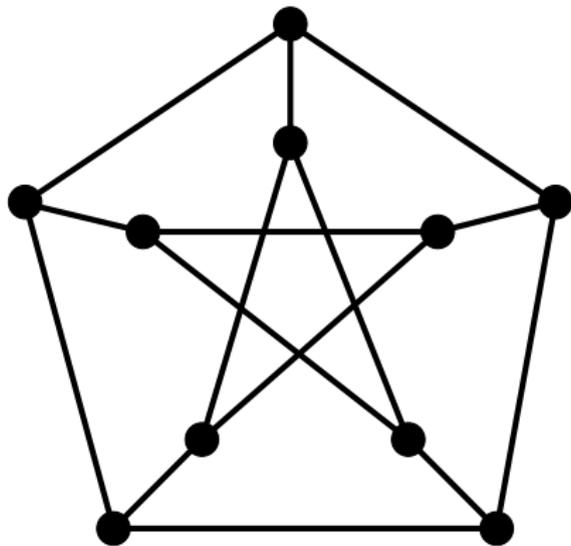


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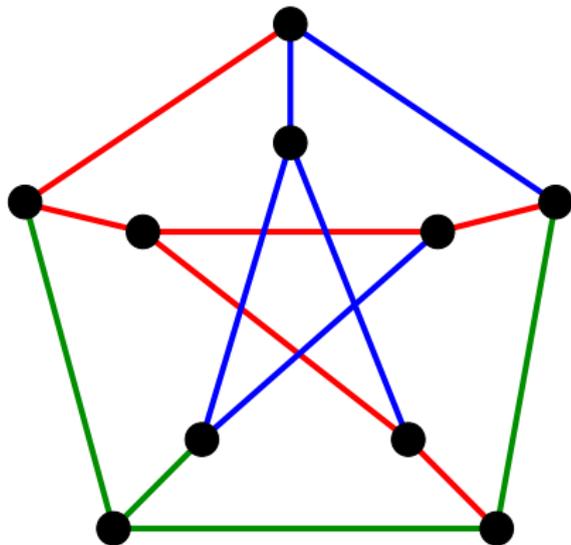
Definitions: *locally irregular edge-colouring*, *irregular chromatic index*

c is *locally irregular* if its every colour class induces a locally irregular subgraph. The *irregular chromatic index* of G , denoted $\chi'_{\text{irr}}(G)$, is the least number of colours in a locally irregular edge-colouring of G (if any).

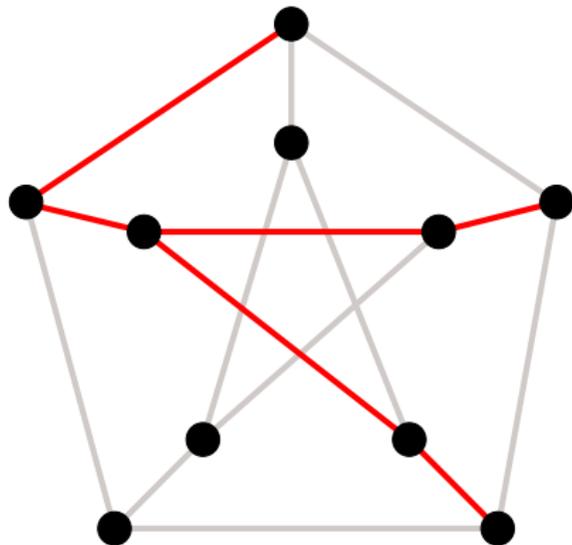
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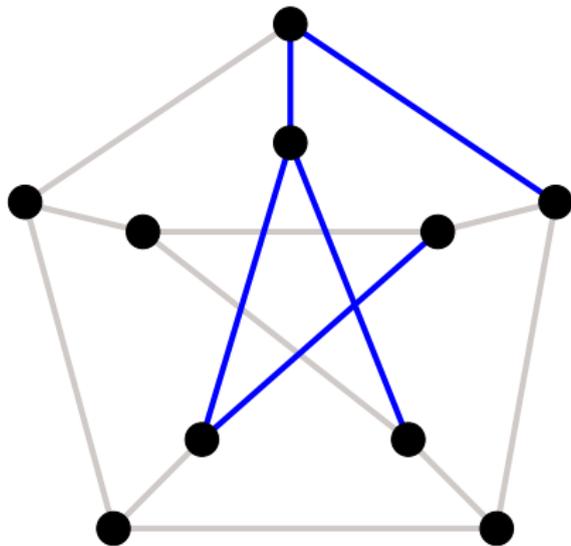
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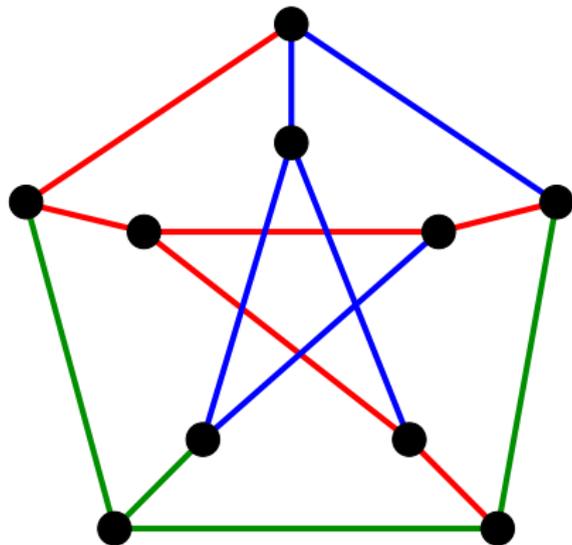
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$$\chi'_{\text{irr}}(\text{Petersen}) \leq 3$$

Locally irregular edge-colouring

$\chi'_{\text{irr}}(G) = \infty$ if G cannot be coloured

Theorem [Baudon, B., Przybyło, Woźniak – 2013+]

$\chi'_{\text{irr}}(G) = \infty$ iff G is an odd length path or cycle, or belongs to a well-characterized family of tree-like graphs made up of triangles.

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- verified for several families of graphs...
- ... including regular graphs of large degree ($\geq 10^7$)

The “Isolated K_2 's” problem

No good upper bound on χ'_{irr} is known

Theorem [Baudon, B., Przybyło, Woźniak – 2013+]

For every graph G with $\chi'_{\text{irr}}(G) \neq \infty$, we have

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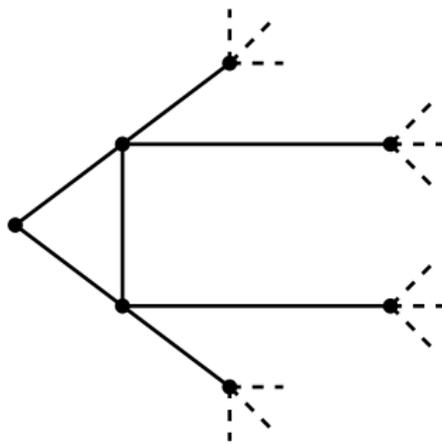
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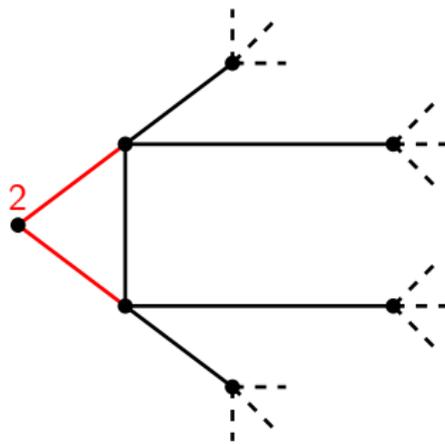
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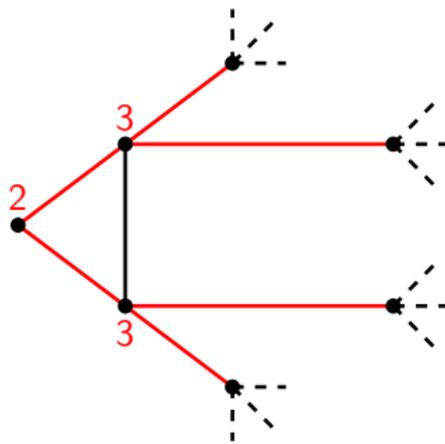
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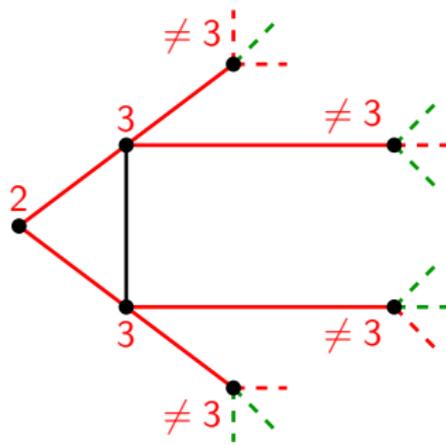
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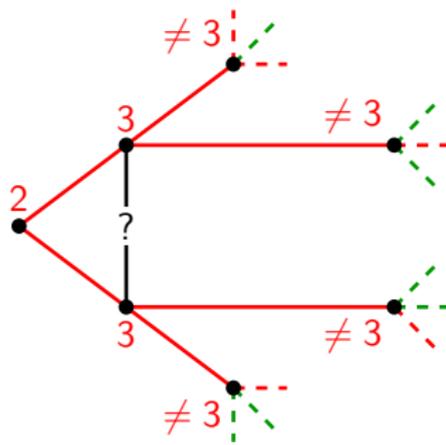
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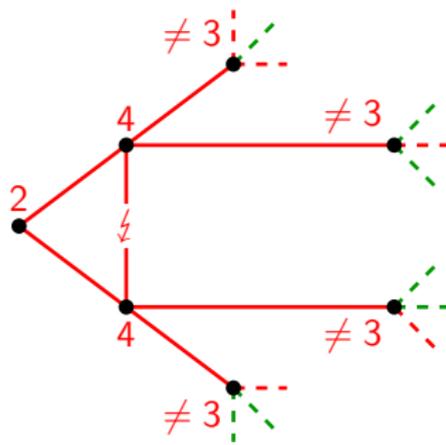
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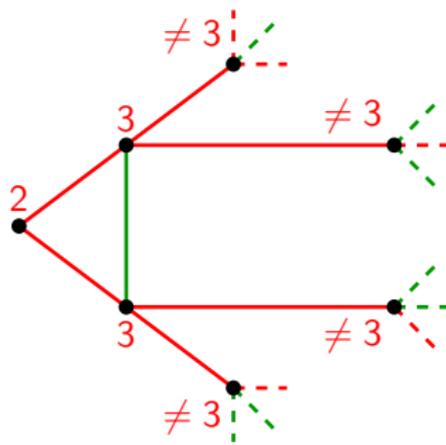
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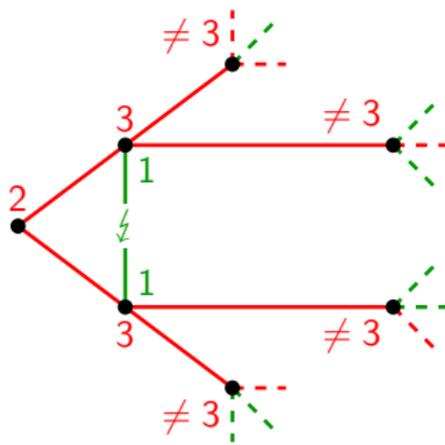
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- locally irregular bipartite graphs ($= 1$)
- complete bipartite graphs (≤ 2)
- regular bipartite graphs with $\delta \geq 3$ (≤ 2)

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... but some of them cannot be decomposed at all

Question [B., Stevens, Woźniak – 2014]

Does it help if we allow “isolated K_2 's” in locally irregular edge-colourings?

Allowing K_2 's

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What if we allow regular components?

!! This problem is different from the original one !!

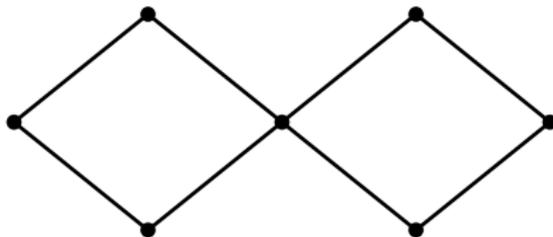
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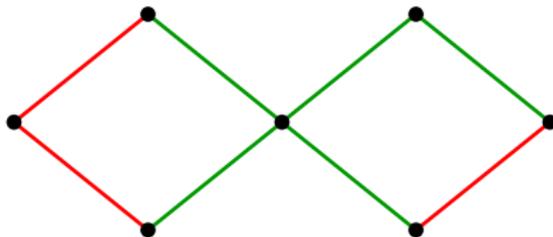
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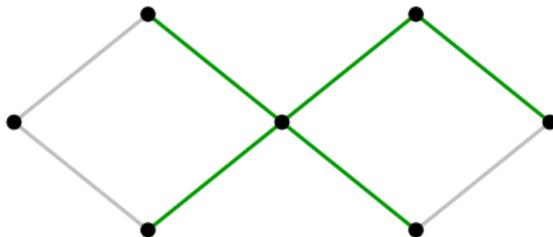
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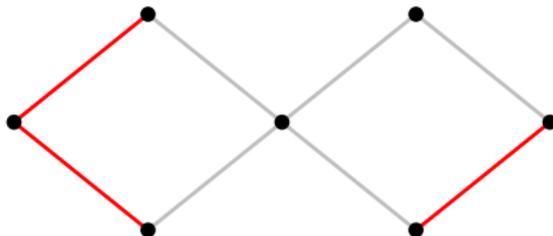
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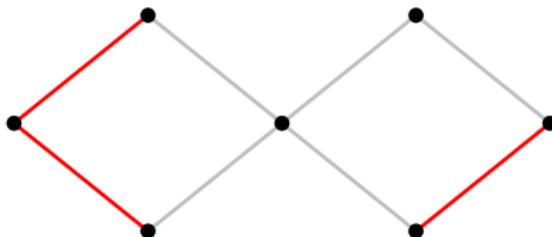
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“uniformization” under using twice as many colours

Regular-irregular chromatic index – easy observations

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The *regular-irregular chromatic index* of G , denoted $\chi'_{\text{reg-irr}}(G)$, is the least number of colours in a regular-irregular edge-colouring of G .

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Conjecture [B., Stevens, Woźniak – 2014]

For every graph G , we have $\chi'_{\text{reg-irr}}(G) \leq 2$.

Bipartite graphs – a constant upper bound

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Otherwise, $G = \text{forest} + \text{Eulerian bipartite}$

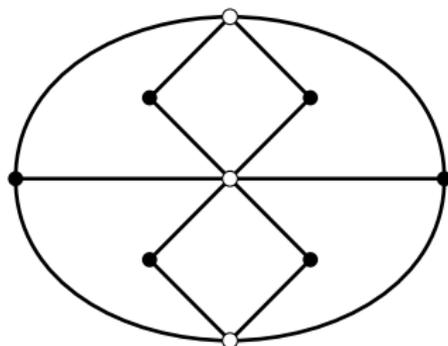
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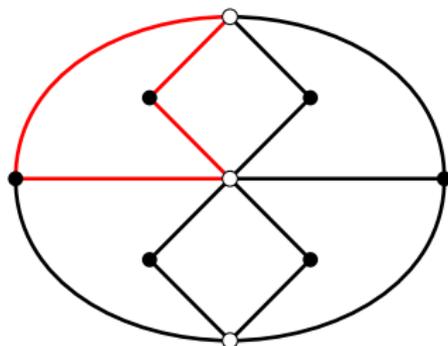
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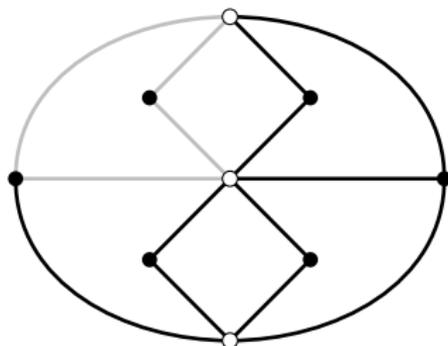
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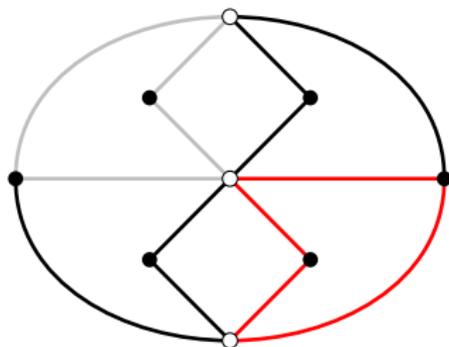
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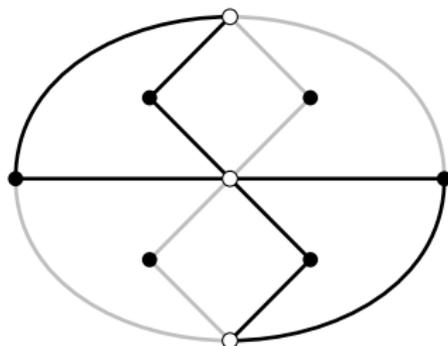
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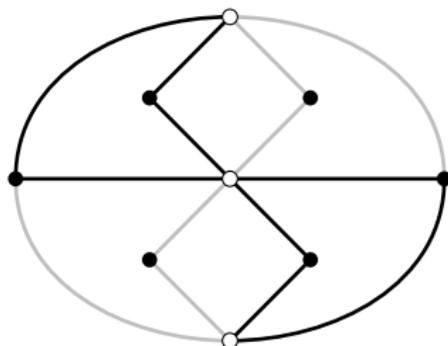
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$$\chi'_{\text{reg-irr}}(G) \leq \chi'_{\text{reg-irr}}(\text{forest}) + \chi'_{\text{reg-irr}}(\text{Eulerian bipartite})$$

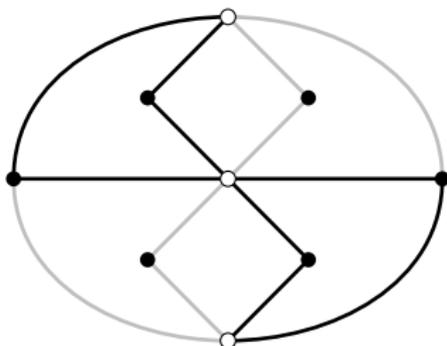
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$$\begin{aligned} \chi'_{\text{reg-irr}}(G) &\leq \chi'_{\text{reg-irr}}(\text{forest}) + \chi'_{\text{reg-irr}}(\text{Eulerian bipartite}) \\ &\leq 2 + 4 \end{aligned}$$

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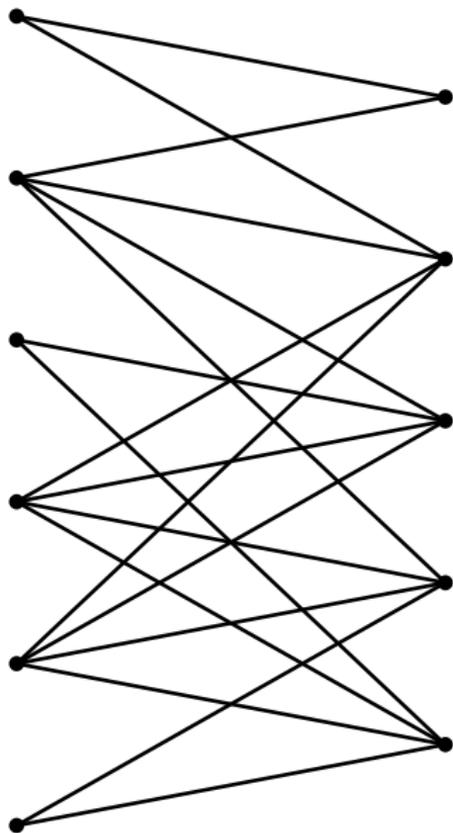
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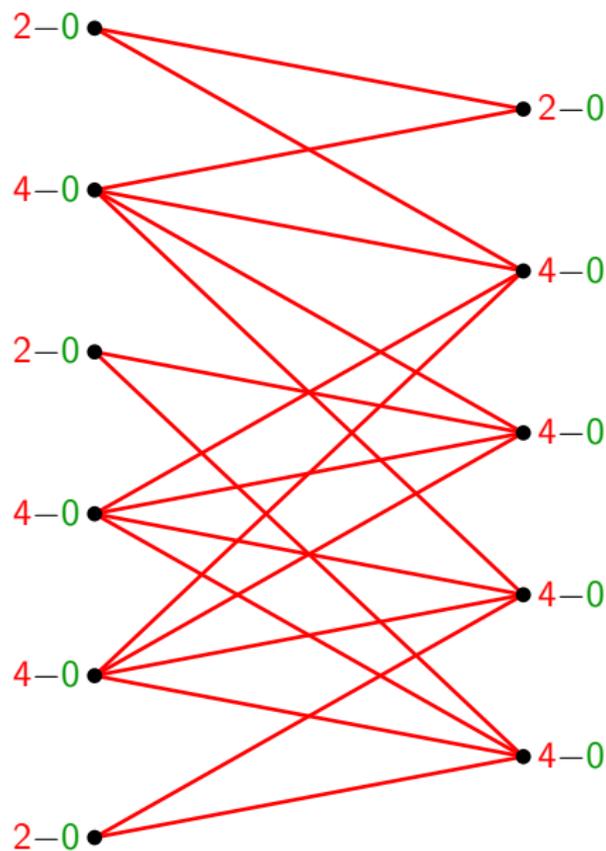
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Start with c consisting of all edges coloured 1. Then switch once the parity of $\#1_c(a)$ for every $a \in A$ by applying the P-S procedure on distinct pairs of vertices of A . ■

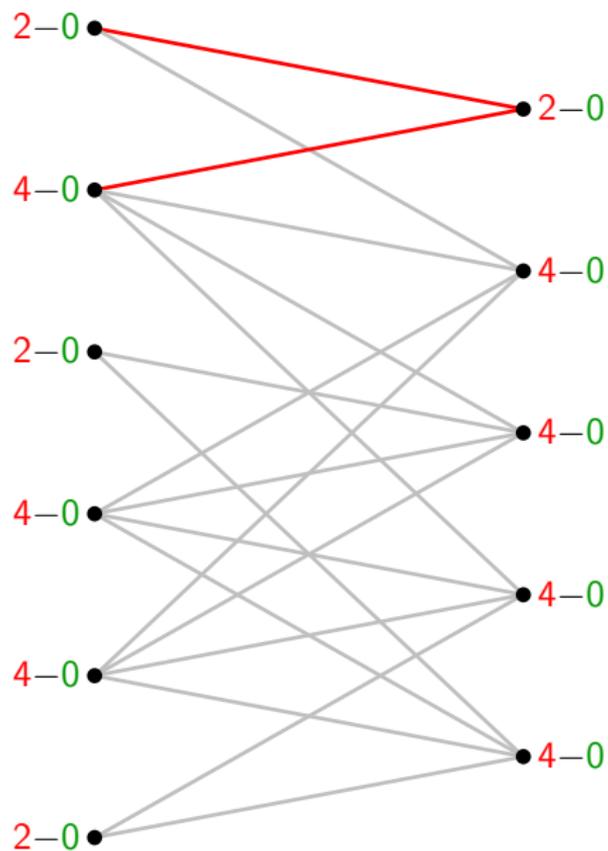
Path-Switching procedure – illustration



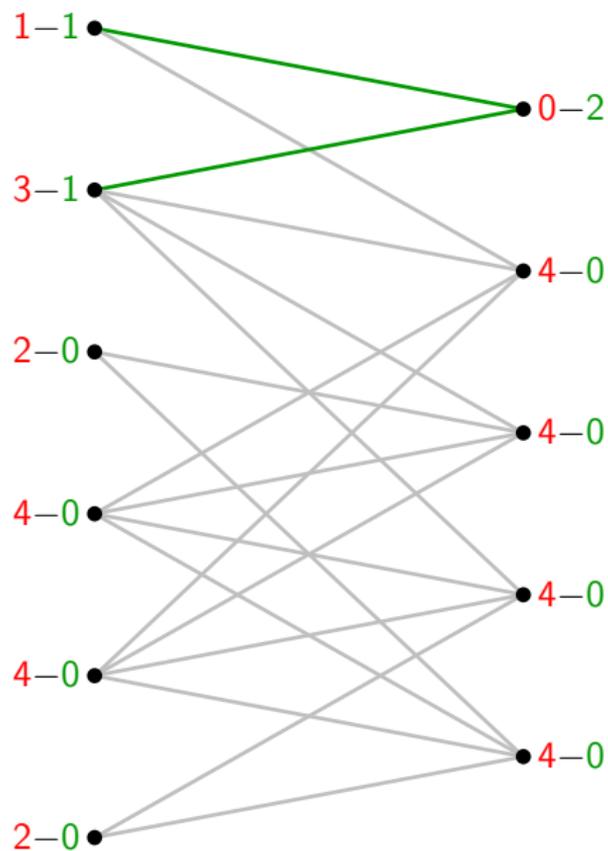
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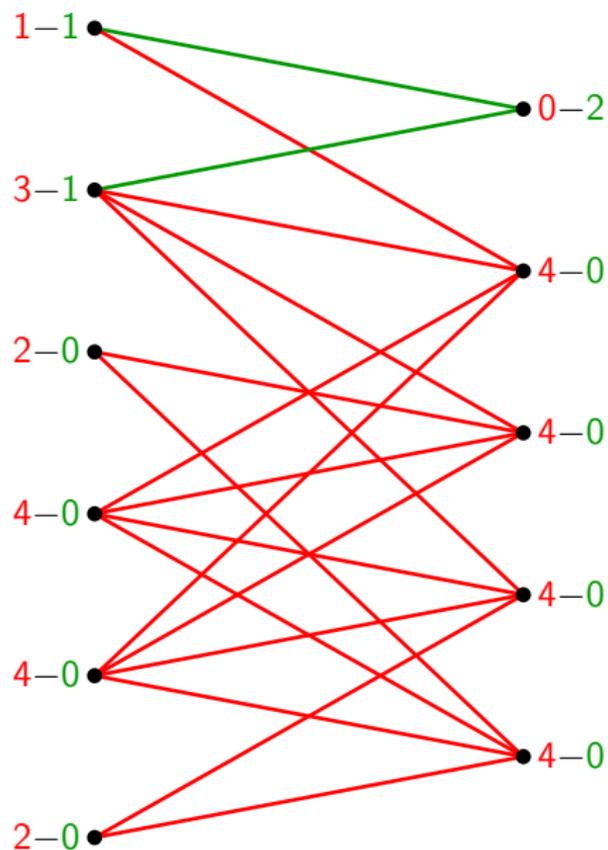
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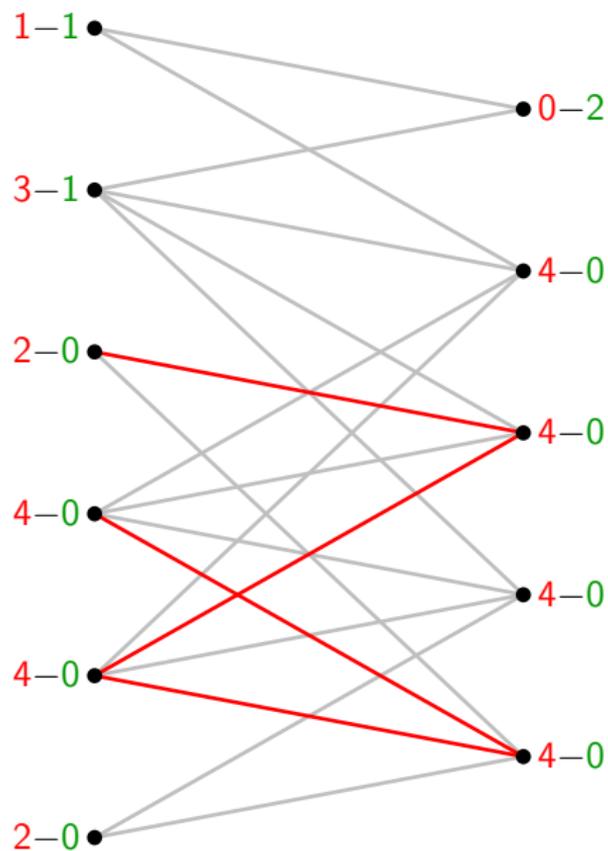
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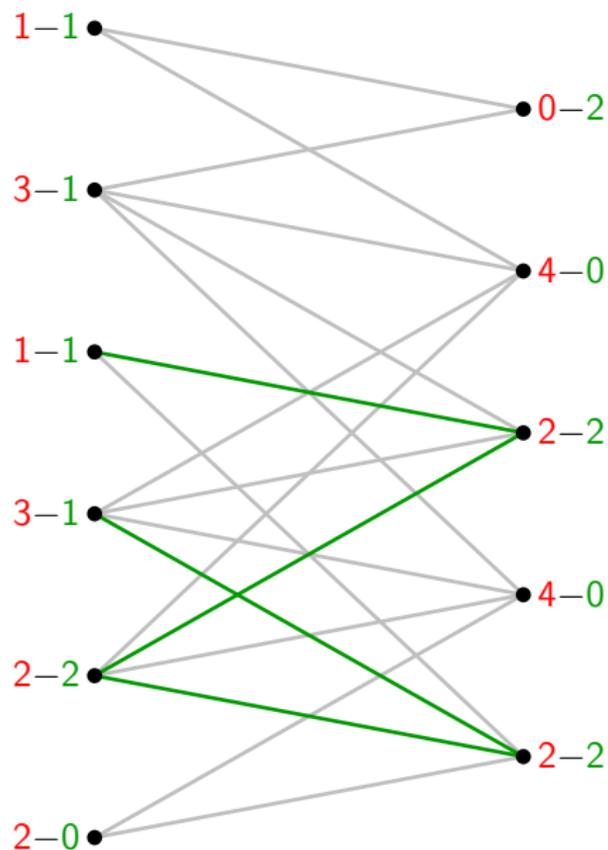
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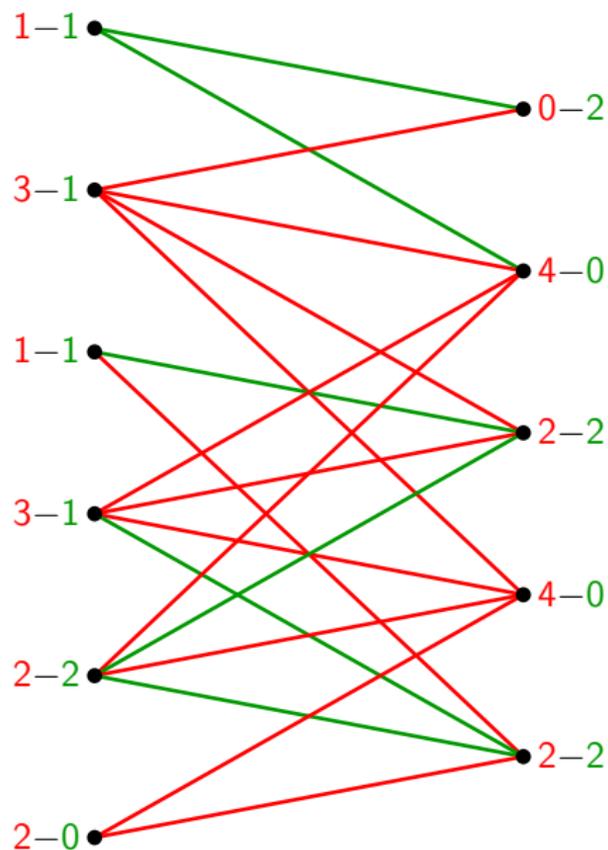
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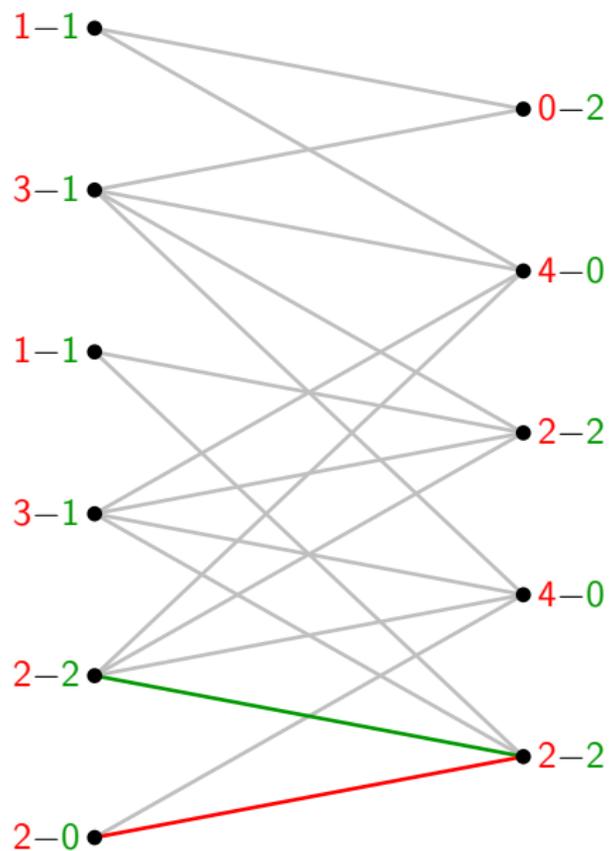
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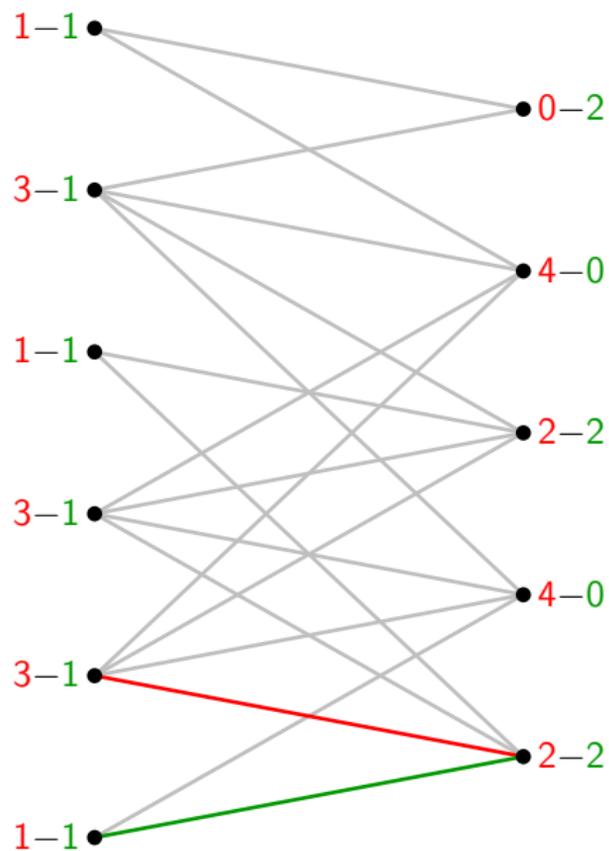
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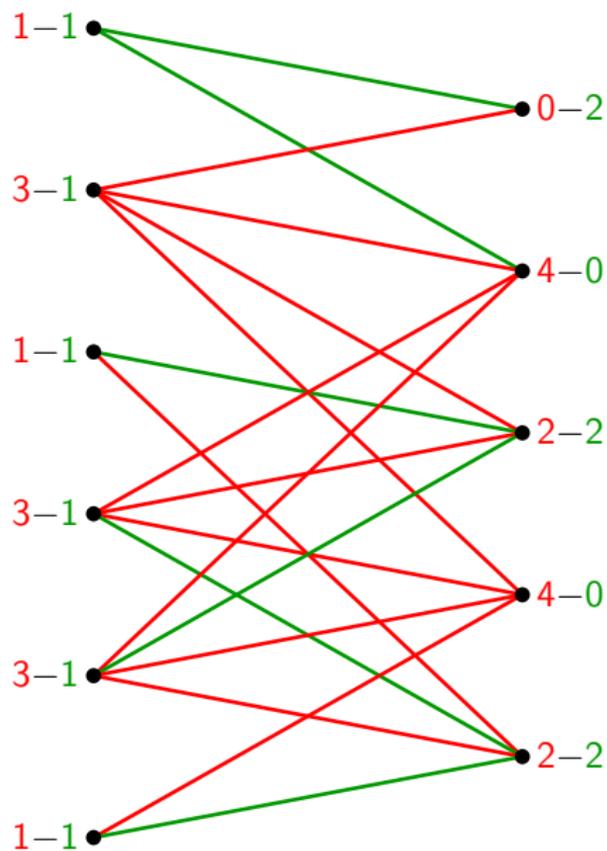
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Actually a can be “conveniently” chosen so that if the obtained 2-edge-colouring is not locally irregular, then it is only because of the maximum component G_1 induced by the edges coloured 1 including those incident to a . But in such a situation the structure of G_1 is so specific that it admits a regular-irregular 3-edge-colouring. So we can just decompose this subgraph separately. ■

All graphs – a result

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For every graph G , we have $\chi'_{\text{reg-irr}}(G) \leq 6 \log_2(\chi(G))$.

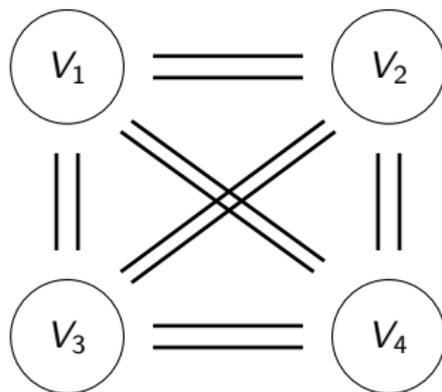
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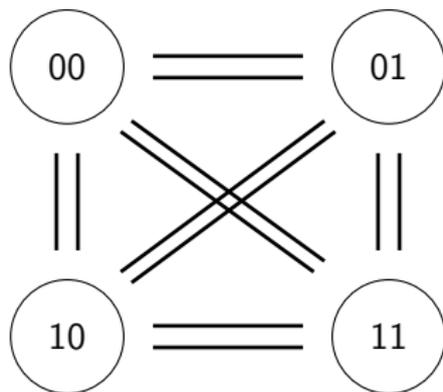
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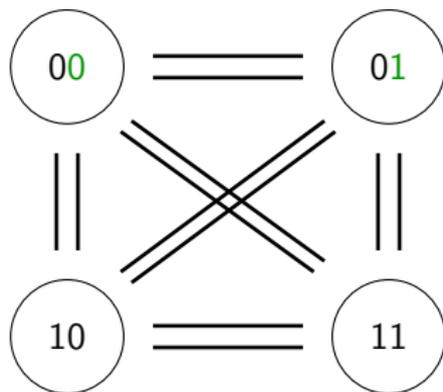
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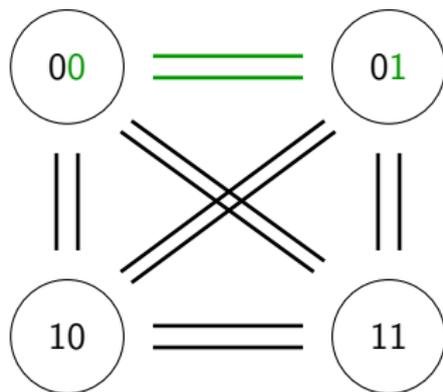
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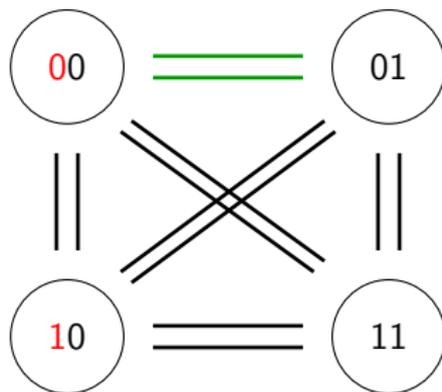
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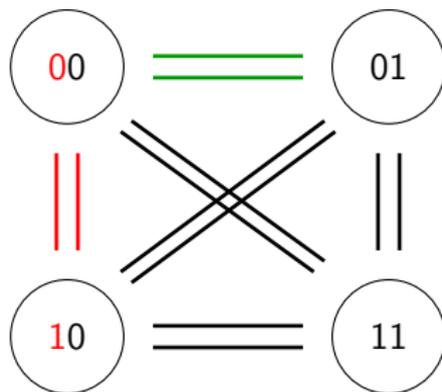
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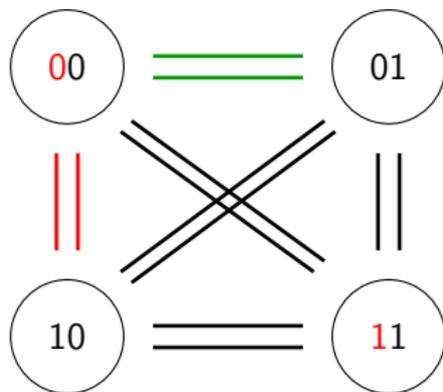
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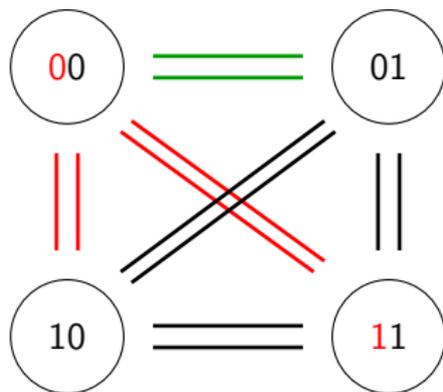
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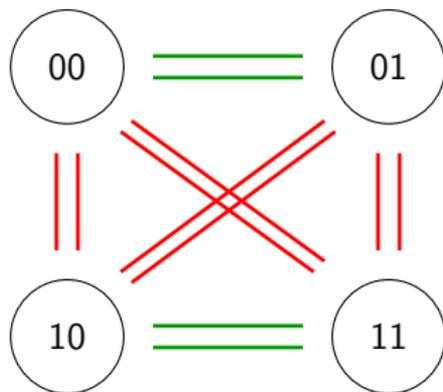
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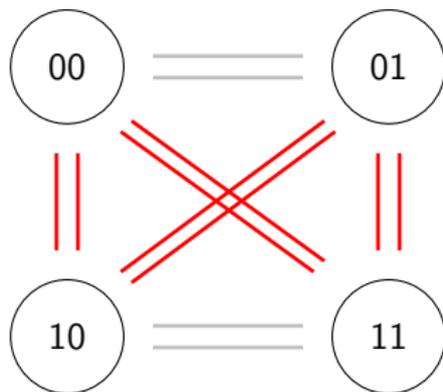
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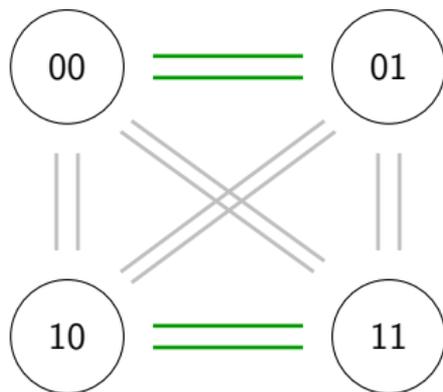
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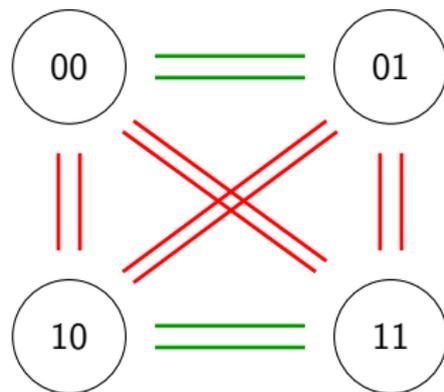
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Then use at most 6 new colors for each of these

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Thank you for your attention.