The complexity of finding orientations with fixed weak diameter

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GT G&A - LaBRI

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Part 1: Oriented distances and diameters

Part 2: 2-weak orientations and oriented chromatic number

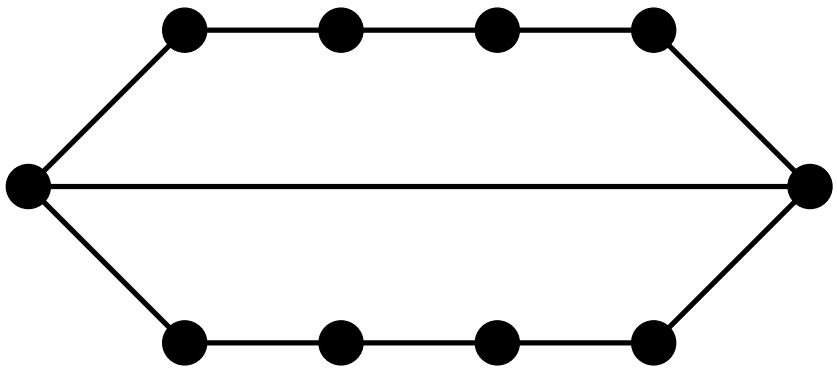
Part 3: Complexity of finding a k-weak orientation

Part 4: Open question

Undirected distance and diameter

Let G be an undirected graph.

The *distance* between u and v in G, denoted $dist(G, u, v)$, is the length of a smallest path starting from u and ending at v. In particular, $dist(G, u, v) = dist(G, v, u)$.



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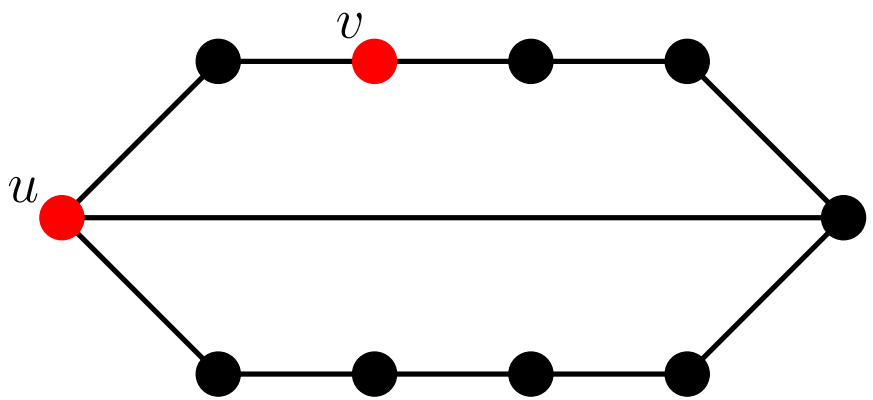

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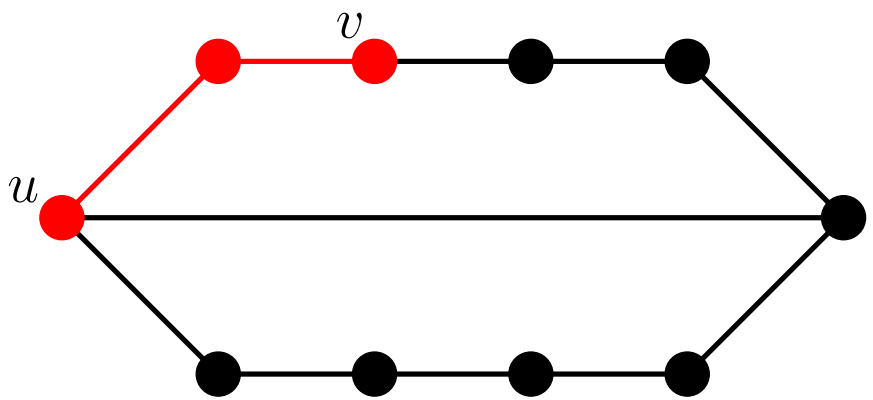

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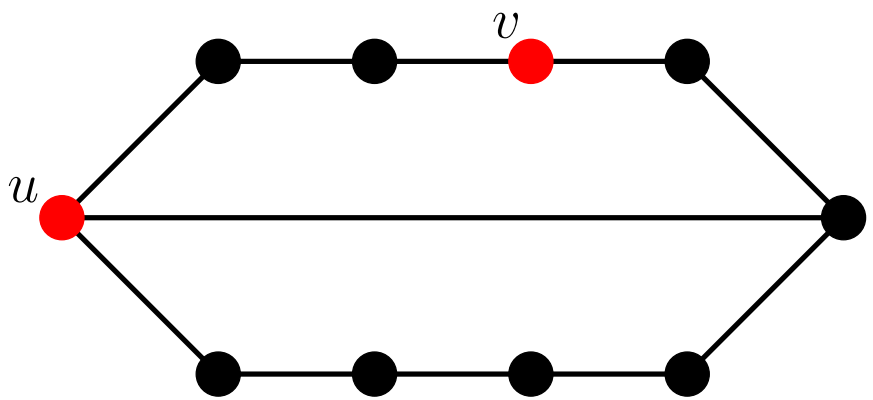

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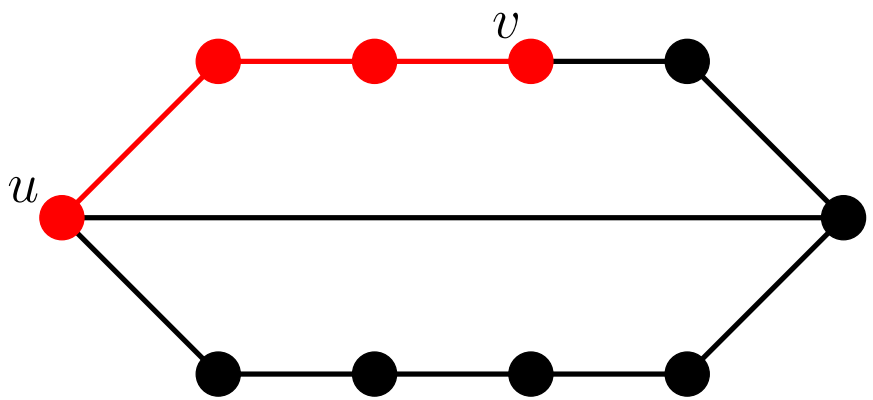

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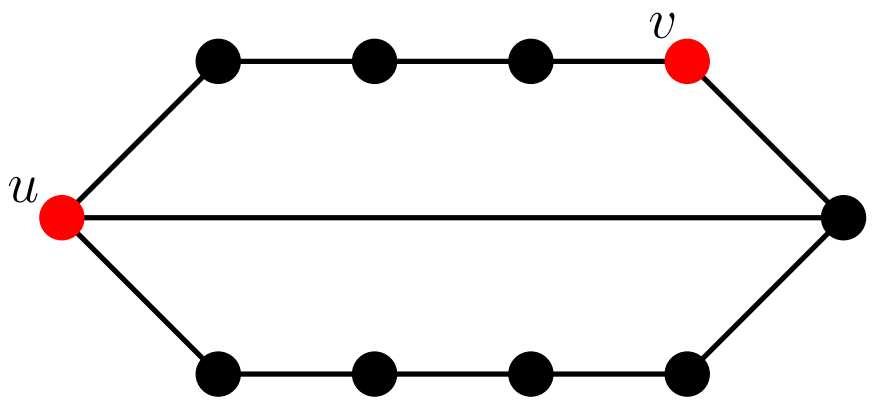

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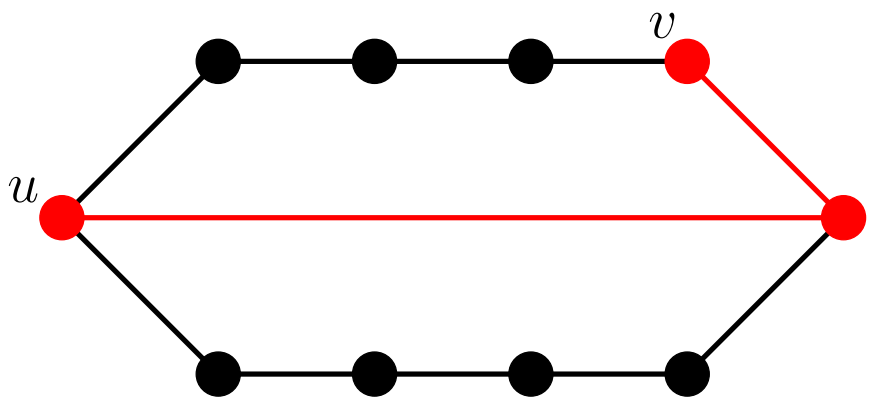

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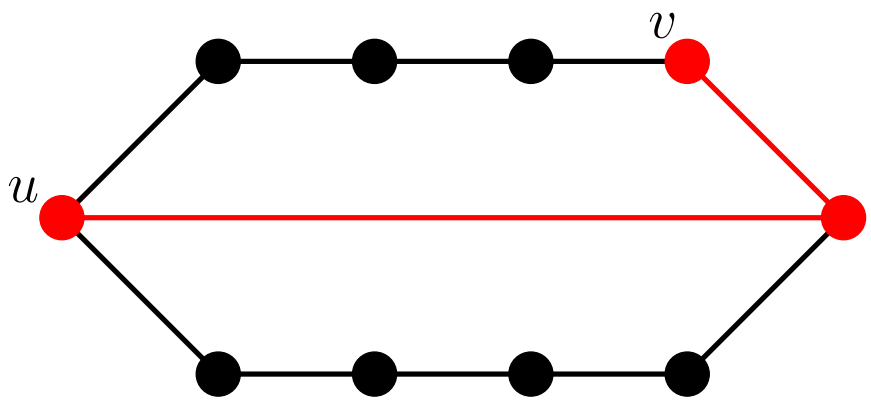

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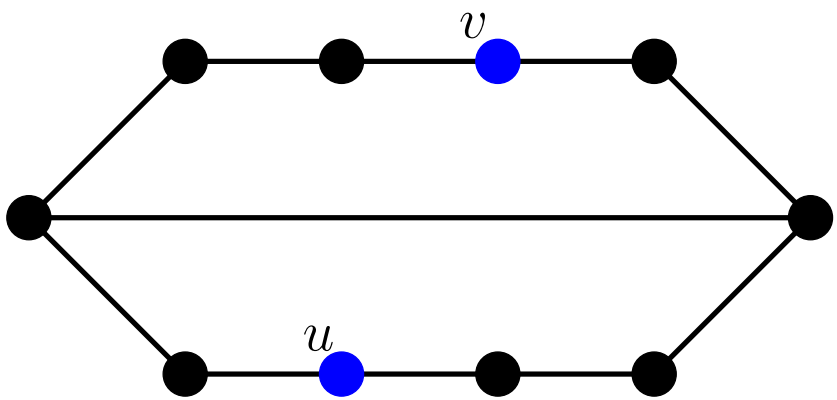

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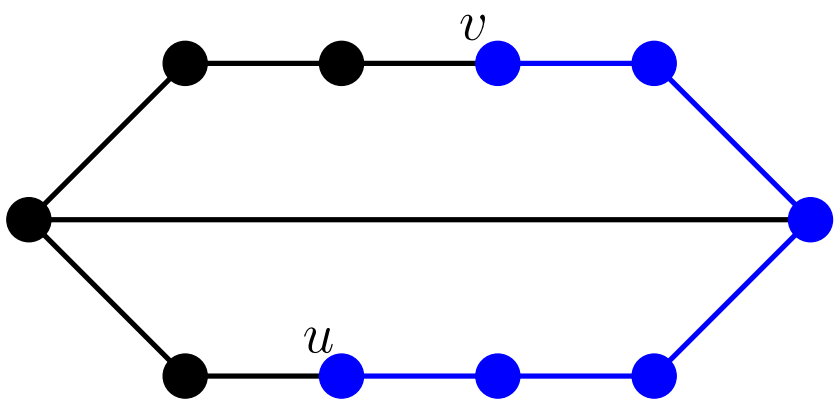

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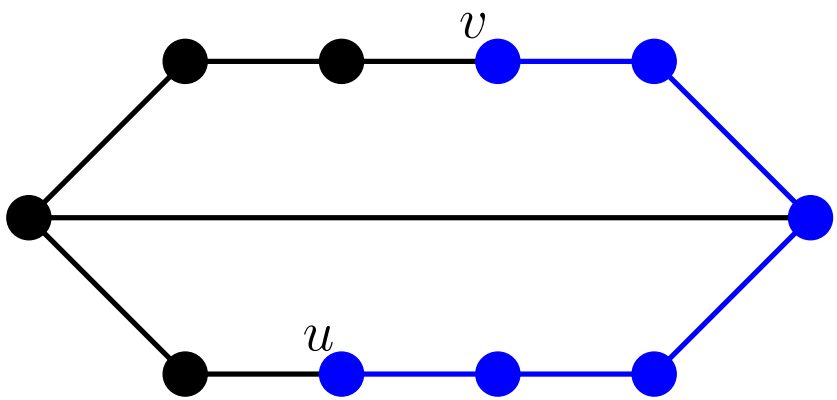

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Oriented distance

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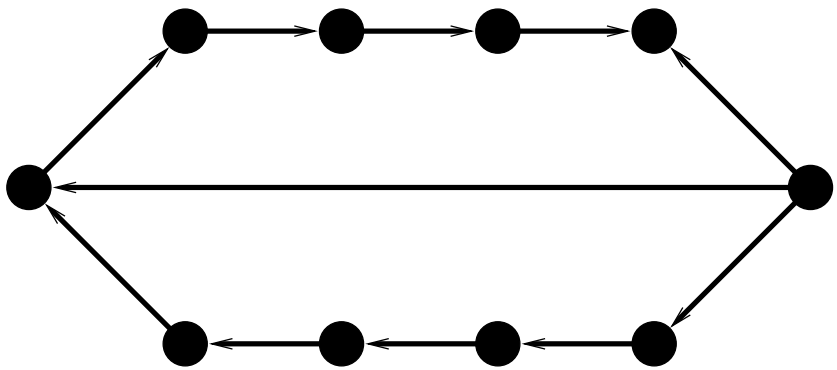

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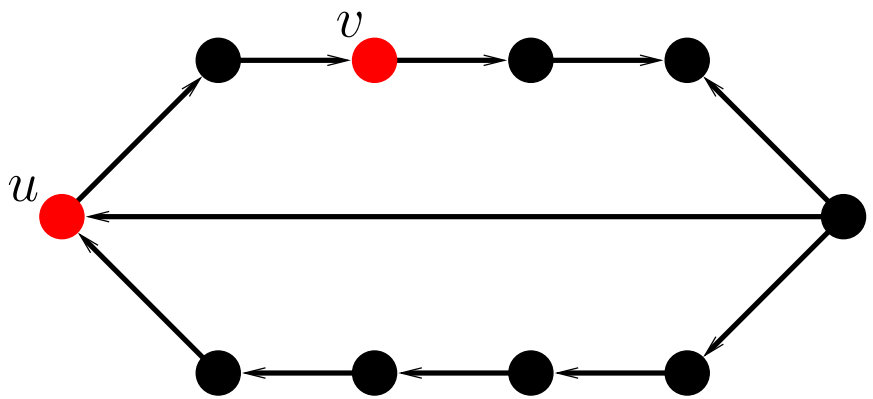

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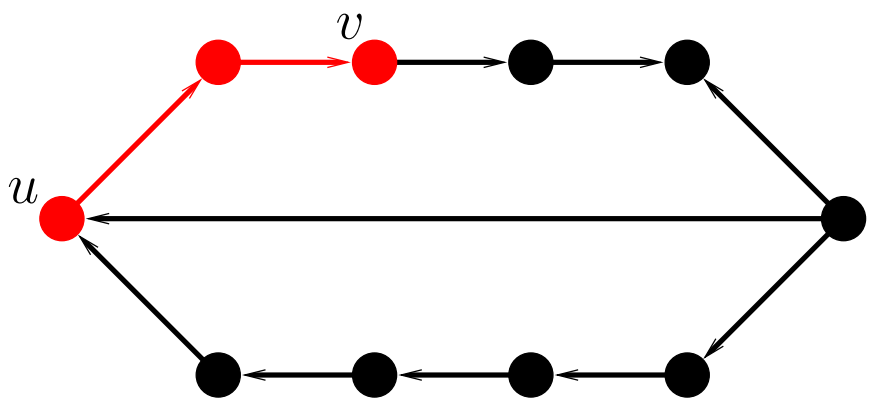

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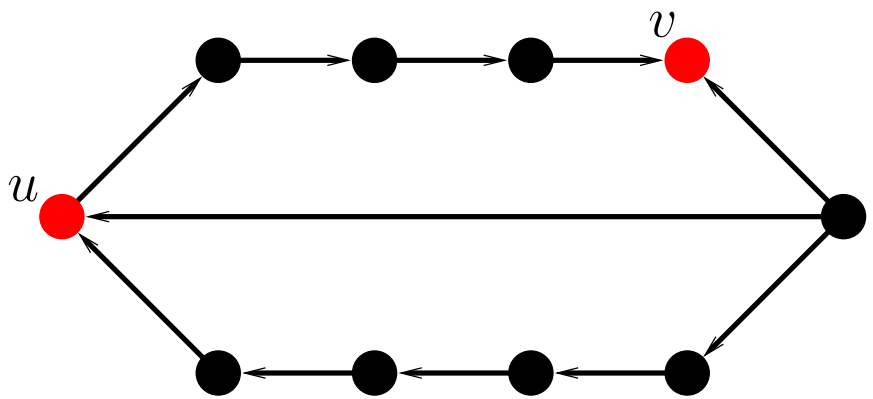

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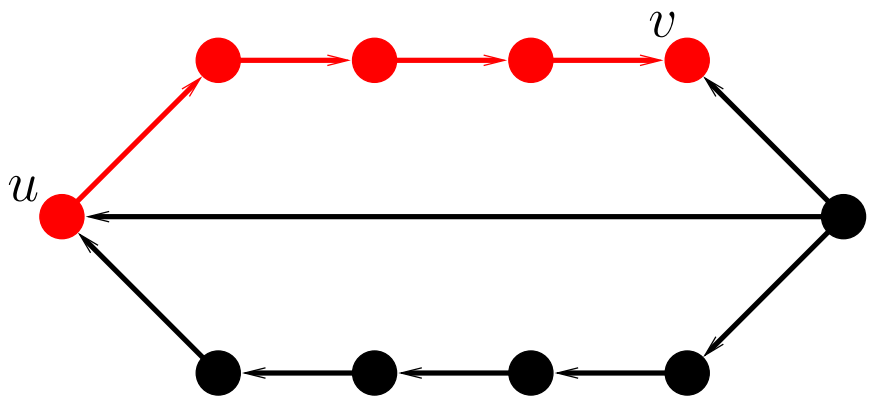

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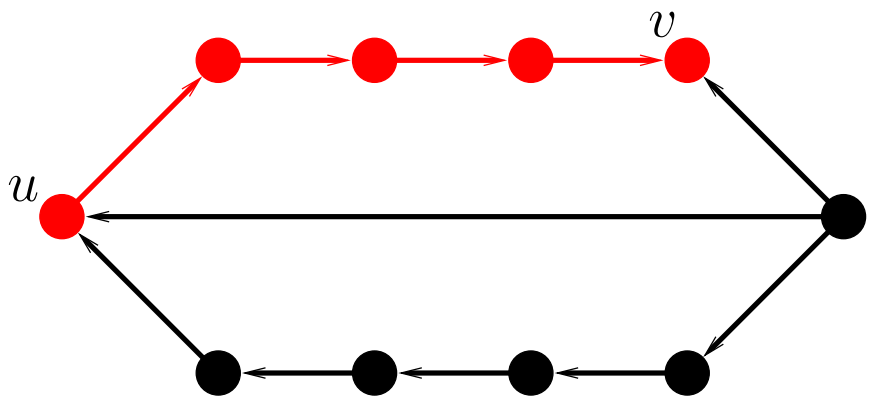
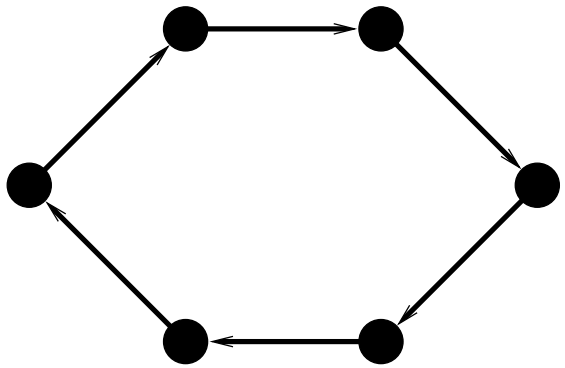


$$dist(\overrightarrow{G}, u, v) = 4$$

Contrary to the undirected case, we may have $dist(\overrightarrow{G}, u, v) \neq dist(\overrightarrow{G}, v, u)$. Therefore, two definitions of an *oriented diameter* can be adopted.

Oriented distances and diameters

The *weak distance* $dist_w(\overrightarrow{G}, u, v)$ between u and v is $\min\{dist(\overrightarrow{G}, u, v), dist(\overrightarrow{G}, v, u)\}$.



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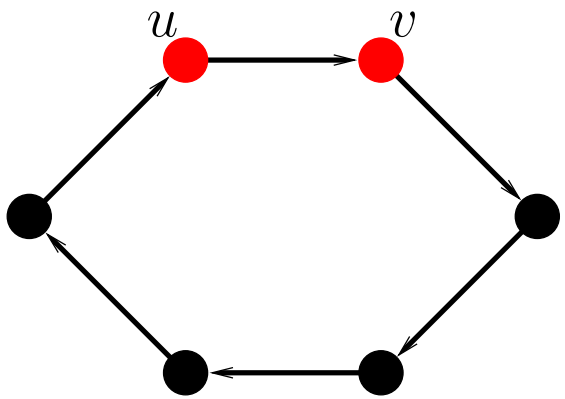

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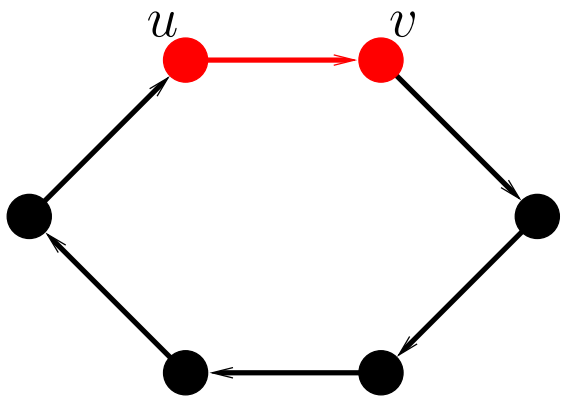

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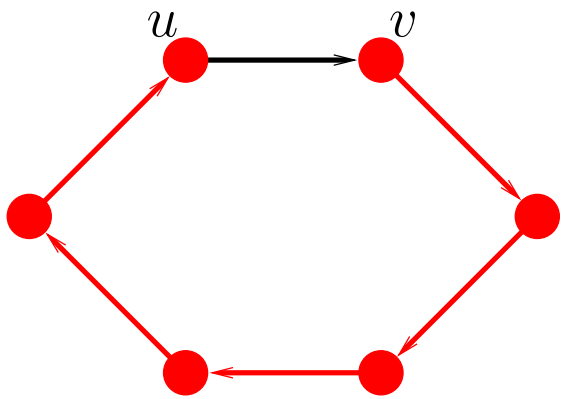

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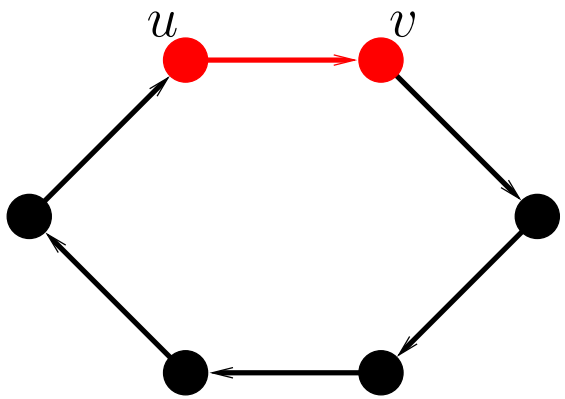

$$dist_w(\overrightarrow{G}, u, v) = 1 = \min\{1, 5\}$$

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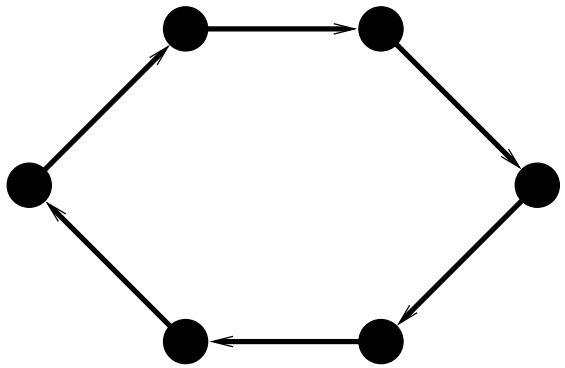

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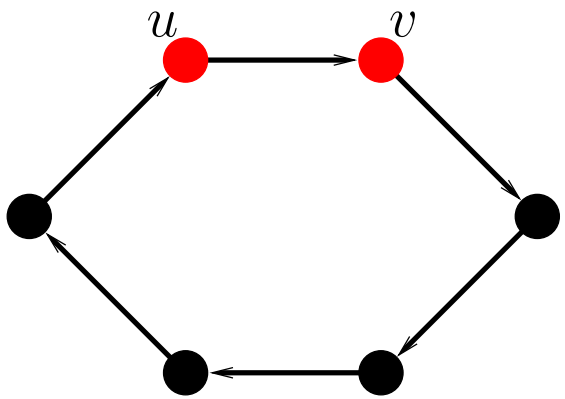

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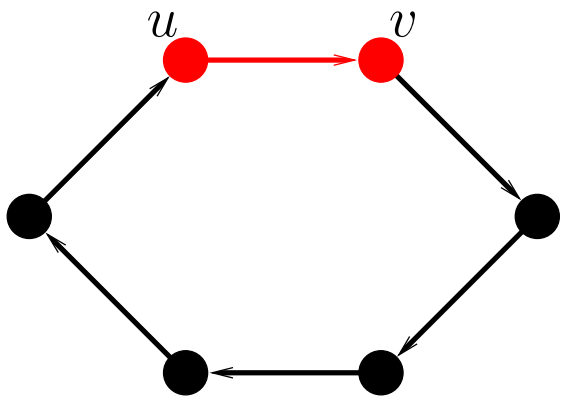

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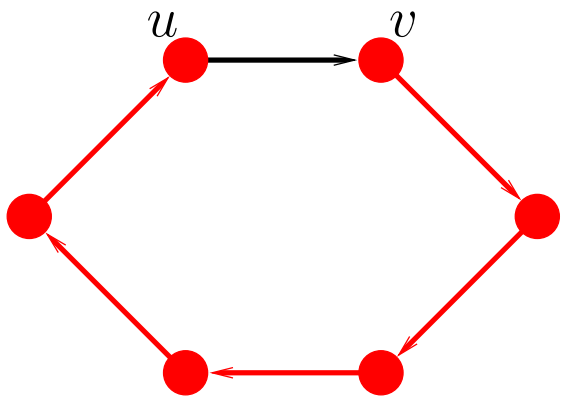

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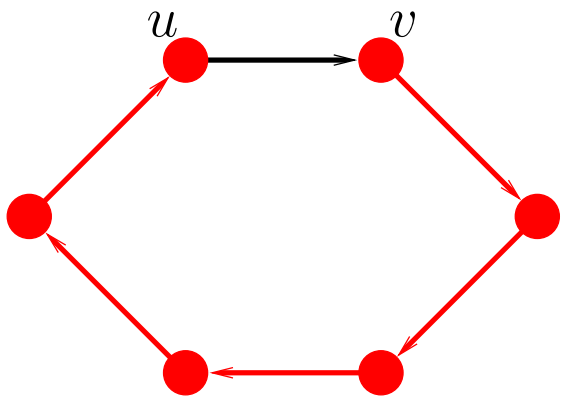

$$dist_s(\overrightarrow{G}, u, v) = 5 = \max\{1, 5\}$$

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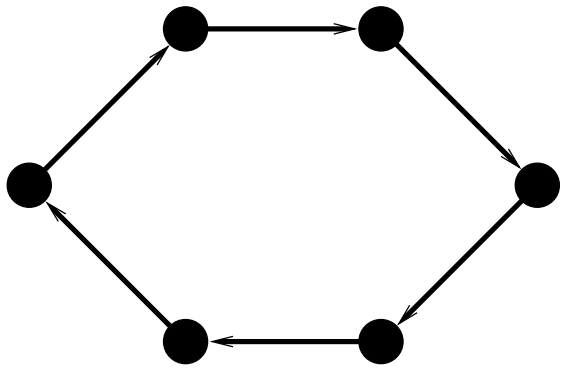

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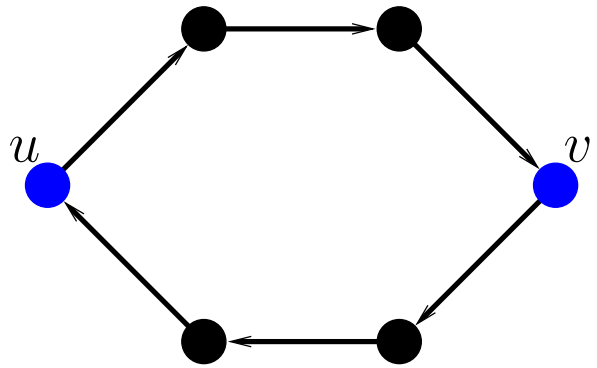

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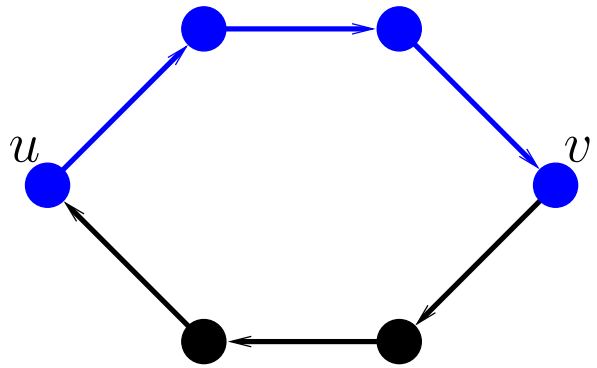

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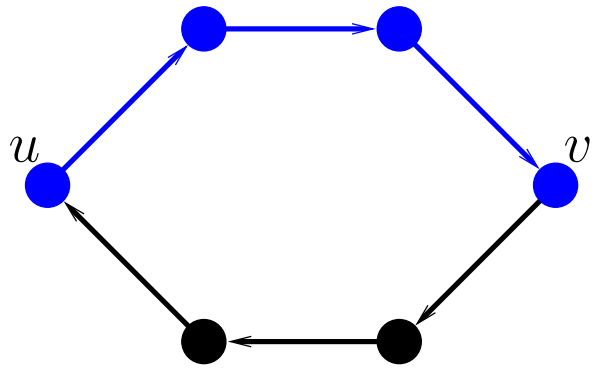

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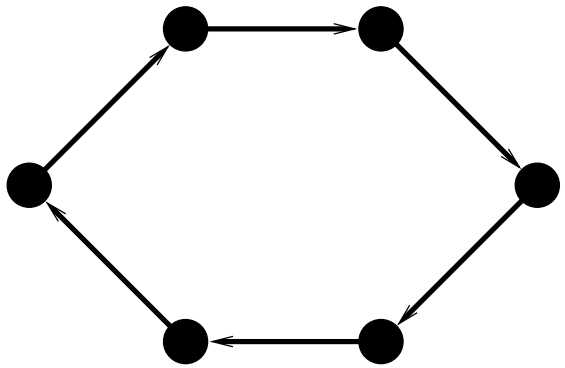

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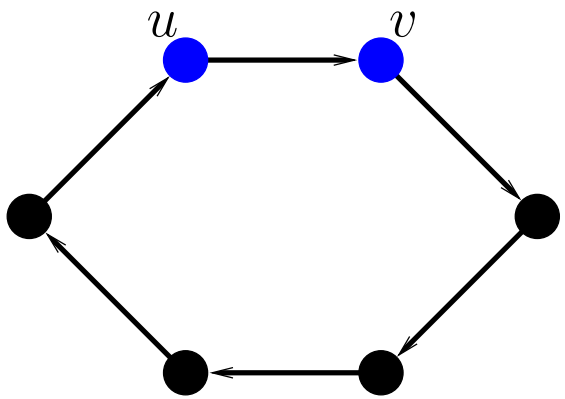

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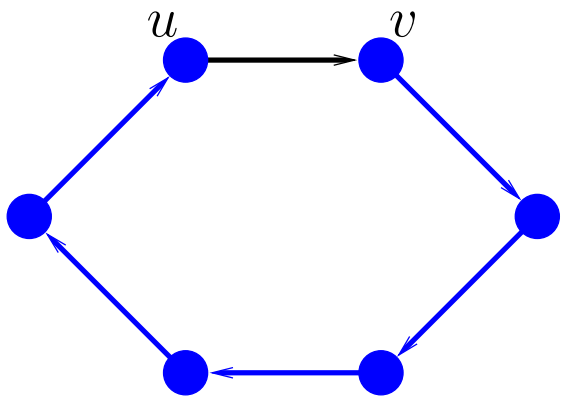

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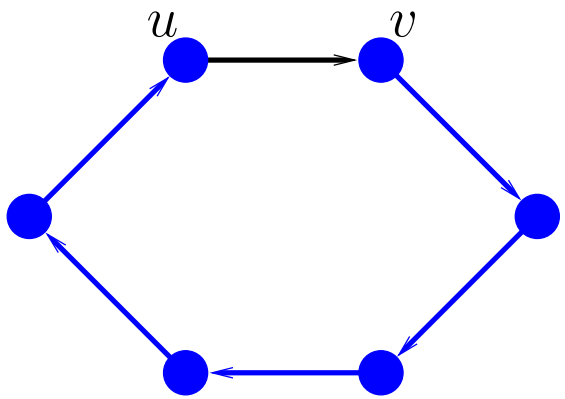

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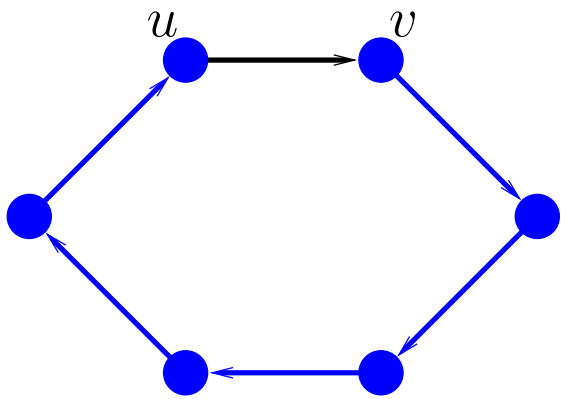

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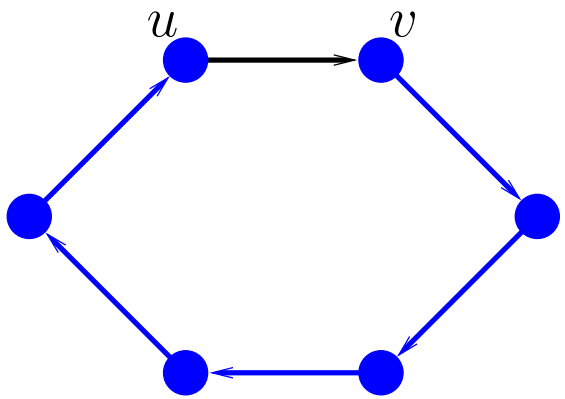

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Let $k \geq diam(G)$. An orientation $\overrightarrow{G}$ of G is *k-weak* (resp. *k-strong*) if $diam_w(\overrightarrow{G}) \leq k$ (resp. $diam_s(\overrightarrow{G}) \leq k$).

Part 1: Oriented distances and diameters

Part 2: 2-weak orientations and oriented chromatic number

Part 3: Complexity of finding a k-weak orientation

Part 4: Open question

Oriented colouring of oriented graphs

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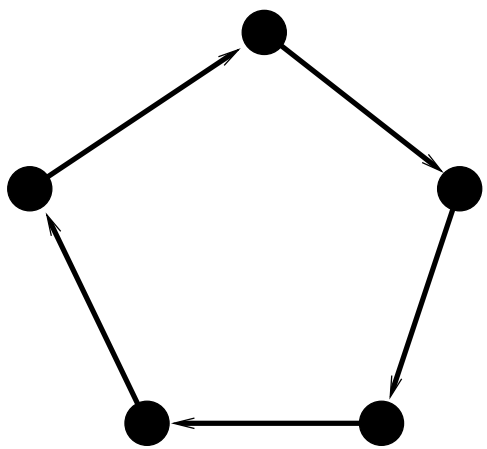

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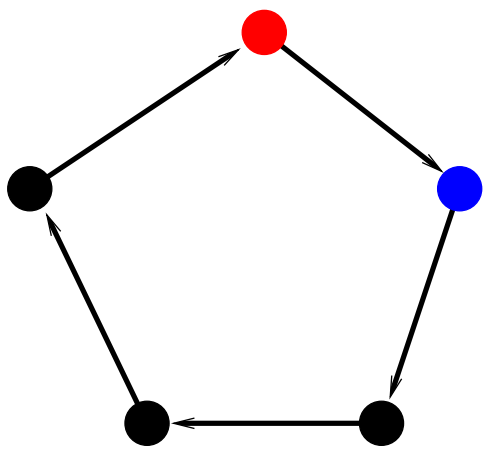

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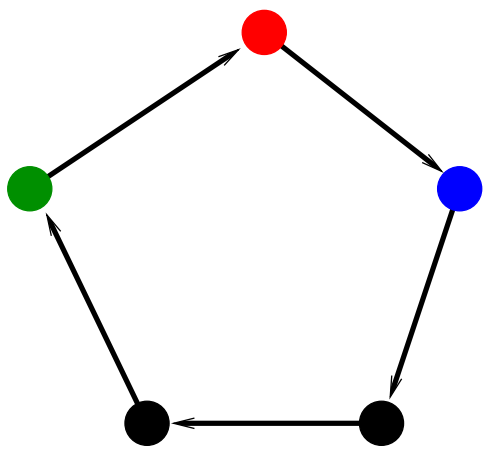

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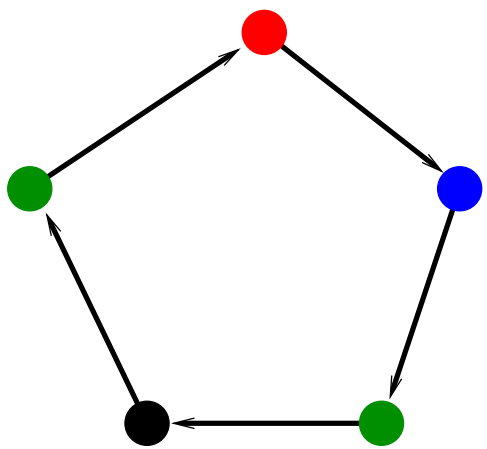

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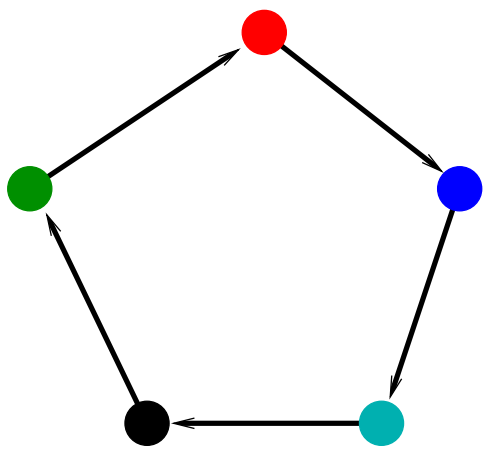

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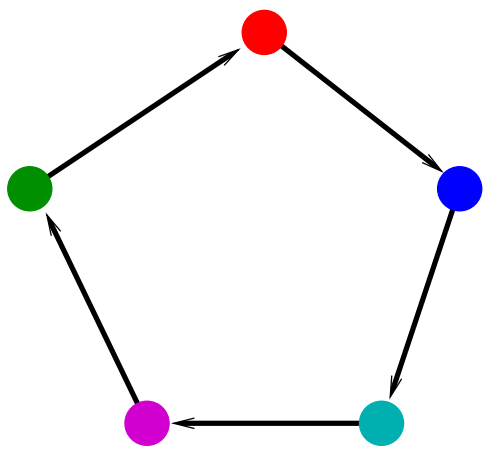

Oriented chromatic number of undirected graphs

An oriented graph $\overrightarrow{G}$ is an *oriented clique* (or *o-clique* for short) if $\chi_o(\overrightarrow{G}) = |V(\overrightarrow{G})|$.

Note that if $\overrightarrow{uvw}$ is a directed path of $\overrightarrow{G}$, then the colours of u, v and w by ϕ are pairwise distinct. Therefore, $\overrightarrow{G}$ is an o-clique iff $diam_w(\overrightarrow{G}) \leq 2$.

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For an undirected graph G, the oriented chromatic number of G is

$$\chi_o(G) = \max\{\chi_o(\overrightarrow{G}), \text{ where } \overrightarrow{G} \text{ is an orientation of } G\}.$$

Regarding the remark above, we have $\chi_o(G) = |V(G)|$ iff G admits a 2-weak orientation.

Part 1: Oriented distances and diameters

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Part 3: Complexity of finding a k-weak orientation

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Complexity of finding a k-weak or k-strong orientation

Deciding whether G admits a 2-strong orientation is NP-complete (Chvátal and Thomassen, 1978). The NP-hardness of the problem was shown by reduction from the 2-colouring problem of hypergraphs of rank 3.

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We here consider the analogous question regarding weak orientations.

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As a corollary, we get the following result.

> Cor. *B., Duvignau, Kirgizov (2013+)*
> Deciding whether $\chi_o(G) = |V(G)|$ is NP-complete.

Complexity of finding a k-weak orientation

Thm. *B., Duvignau, Kirgizov (2013+)*

Deciding whether G admits a k-weak orientation is NP-complete for every $k \geq 2$.

Proof. Let $k \geq 2$ be fixed. Given an orientation $\overrightarrow{G}$ of G, one can check, for every pair $\{u, v\}$ of vertices, whether there is a directed path with length at most k from u to v in either direction. This can be done in $O(|V(G)|^{O(k)})$. Therefore, the problem is in NP.

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The NP-hardness of the problem is shown by reduction from a variant of 3SAT.

Not-All-Equal 3SAT - NAE-3SAT

Instance: A 3CNF formula F over clauses $C_1, ..., C_m$ and variables $x_1, ..., x_n$.

Question: Is F *nae-satisfiable*, i.e. satisfiable in such a way that none of its clauses has its three literals receiving the same truth value?

In particular, we consider the **monotone** version of NAE-3SAT, i.e. where no clause of F contains a negated variable, which remains NP-complete.

Complexity of finding a k-weak orientation

Thm. *B., Duvignau, Kirgizov (2013+)*

Deciding whether G admits a k-weak orientation is NP-complete for every $k \geq 2$.

We may assume that F does not contain a clause whose three variables are the same.

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We may assume that F does not contain a clause whose three variables are the same.

From F, we construct, in polynomial time, a graph G_F such that

$$F \text{ is nae-satisfiable} \Leftrightarrow G_F \text{ admits a } k\text{-weak orientation.}$$

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$$F \text{ is nae-satisfiable} \Leftrightarrow G_F \text{ admits a } k\text{-weak orientation.}$$

We first construct the *core* G_F^c of G_F, whose role is to "encode" the structure of F. In particular, from the existence of a *good* orientation of G_F^c, we can deduce a truth assignment that makes F nae-satisfied, and vice-versa. More precisely, an orientation of G_F^c is good if some *representative* pairs of its vertices are at weak distance k.

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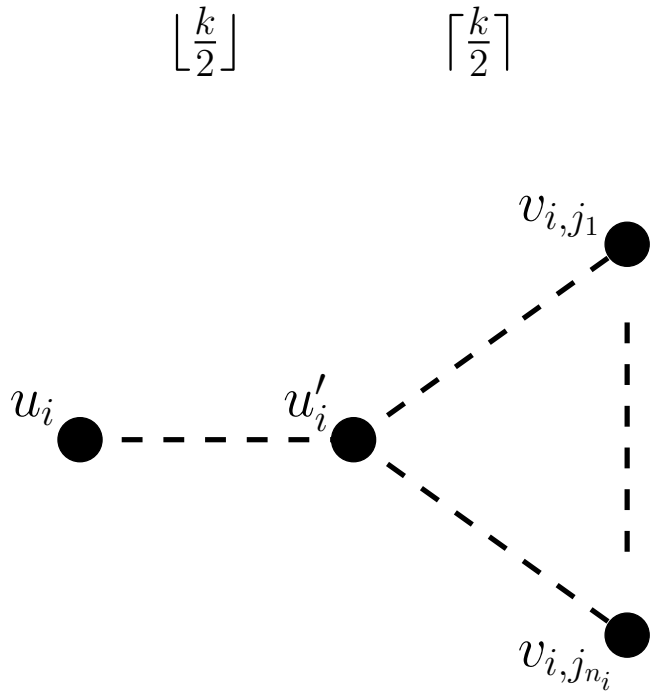
The core G_F^c does not have diameter k, and thus does not admit a k-weak orientation. Therefore, G_F is obtained by adding some structures to G_F^c. This is done in such a way that no new path with length at most k joining two vertices composing a representative pair arose from the construction. In this way, the existence of a k-weak orientation of G_F relies on the existence of a good orientation of G_F^c.

Complexity of finding a k-weak orientation

Thm. *B., Duvignau, Kirgizov (2013+)*

Deciding whether G admits a k-weak orientation is NP-complete for every $k \geq 2$.

For every variable x_i, we add a *variable gadget*, where $j_1, ..., j_{n_i}$ are the distinct indices of the clauses that contain x_i.



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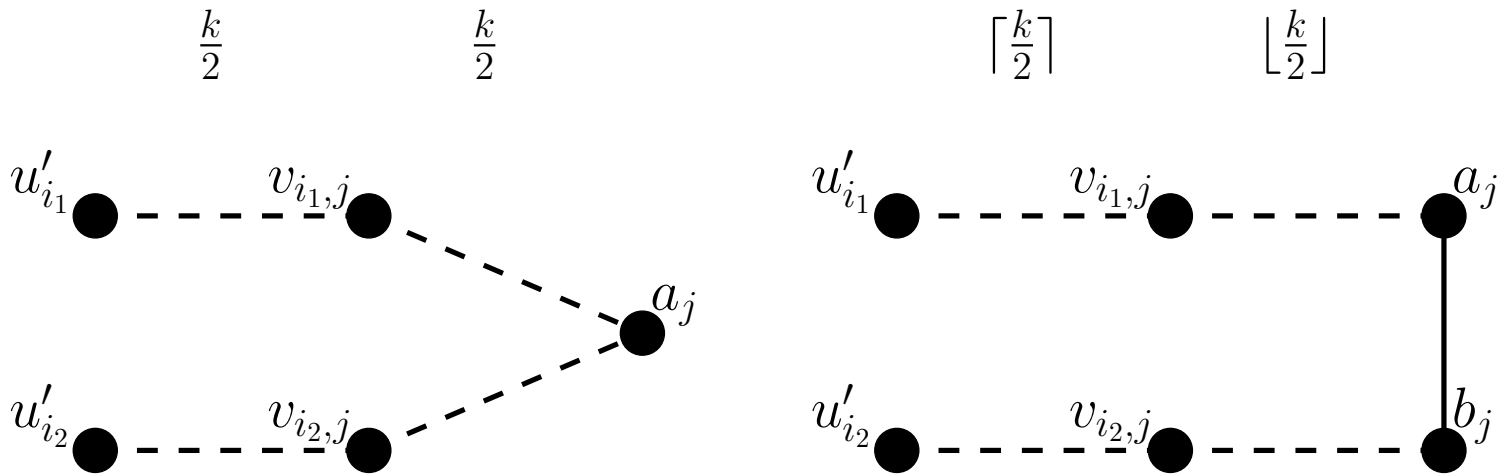
We then associate a *clause gadget* with every clause C_j. First, if C_j has only two distinct variables x_{i_1} and x_{i_2}, then add the following. The clause vertex a_j, b_j or c_j at distance $\lfloor \frac{k}{2} \rfloor$ from $v_{i,j}$ is denote $s(v_{i,j})$.



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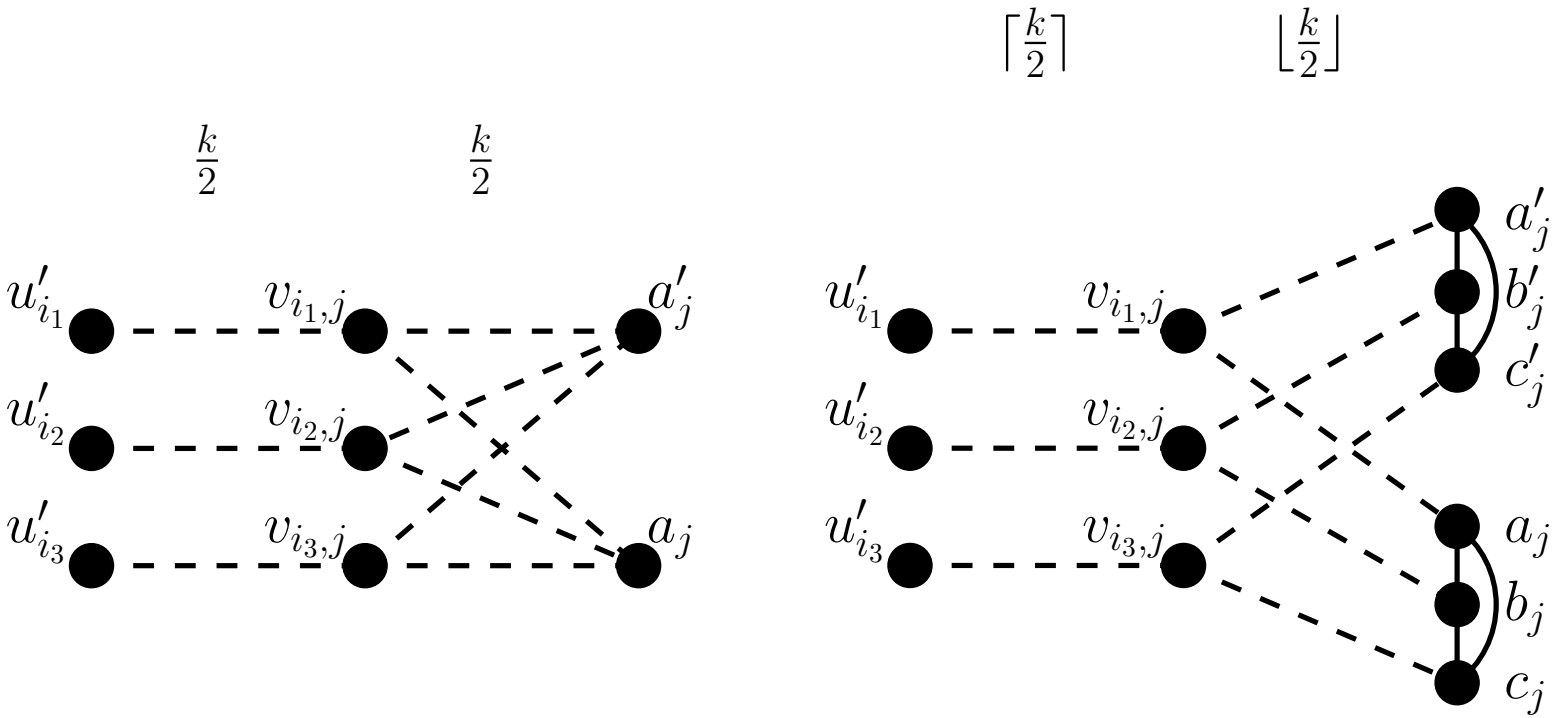
Now, if $C_j = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$, then add the following.



Complexity of finding a k-weak orientation

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Deciding whether G admits a k-weak orientation is NP-complete for every $k \geq 2$.

The representative pairs of G_F^c are those of the form:

1. $\{u_i, v_{i,j}\}$ where $i \in \{1, ..., n\}$, $j \in \{1, ..., m\}$, and $x_i \in C_j$.
2. $\{u'_i, s(v_{i,j})\}$ where $i \in \{1, ..., n\}$, $j \in \{1, ..., m\}$, and $x_i \in C_j$.
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Because $u_i ... u'_i ... v_{i,j_1}$ is the only path with length at most k from u_i to v_{i,j_1}, a good orientation of G_F^c includes $\overrightarrow{u_i ... u'_i ... v_{i,j_1}}$ wlog. Since $\overrightarrow{u_i ... u'_i}$, we also have $\overrightarrow{u'_i ... v_{i,j_2}}, ..., \overrightarrow{u'_i ... v_{i,j_{n_i}}}$ so that u_i and $v_{i,j_2}, ..., v_{i,j_{n_i}}$ are at weak distance k. Similarly, because $u'_i ... v_{i,j_1} ... s(v_{i,j_1})$, ..., $u'_i ... v_{i,j_{n_i}} ... s(v_{i,j_{n_i}})$ are the only paths with length at most k from u'_i to the $s(v_{i,j})$'s, we have $\overrightarrow{v_{i,j_1} ... s(v_{i,j_1})}$,..., $\overrightarrow{v_{i,j_{n_i}} ... s(v_{i,j_{n_i}})}$.

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If x_i is set to true (resp. false), then it gives a 1 (resp. 0) to each clause that contains x_i.

Complexity of finding a k-weak orientation

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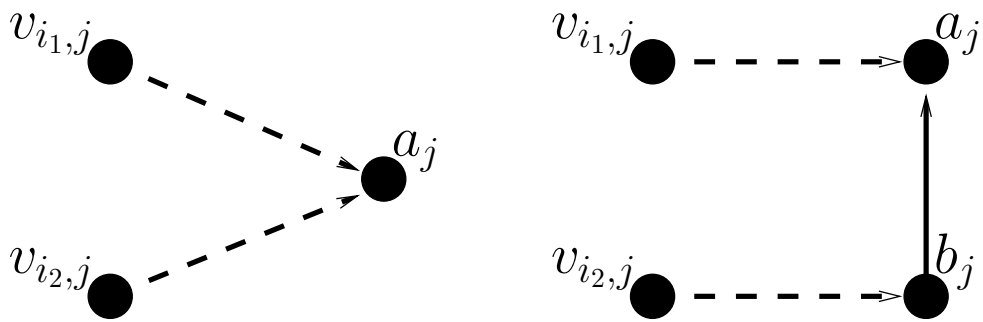
Besides, the paths from the $v_{i,j}$'s to the $s(v_{i,j})$'s cannot all have the same direction.



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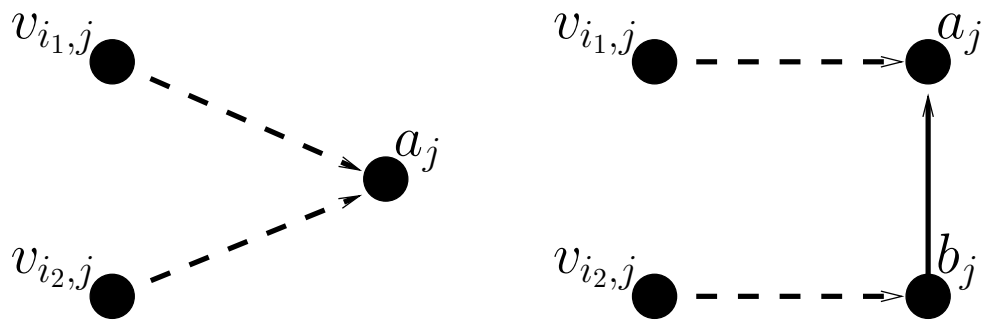

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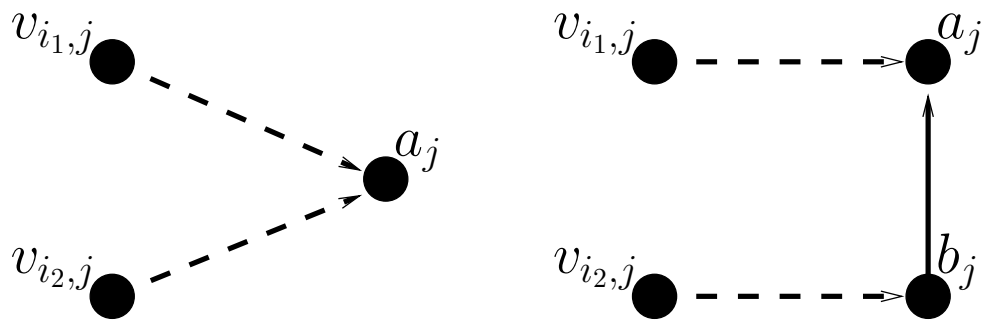

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Having $\overrightarrow{u_i \ldots u_i'}$ in an orientation of G_F^c (resp. $\overrightarrow{u_i' \ldots u_i}$) is like setting x_i to true (resp. false). Hence, we have an equivalence between nae-satisfying F and finding a good orientation of G_F^c.

Complexity of finding a k-weak orientation

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Deciding whether G admits a k-weak orientation is NP-complete for every $k \geq 2$.

We now modify G_F so that there is a partial orientation of its edges such that, for every pair $\{u, v\}$ which is not representative, the weak distance between u and v is at most k. This is done without altering the equivalence.

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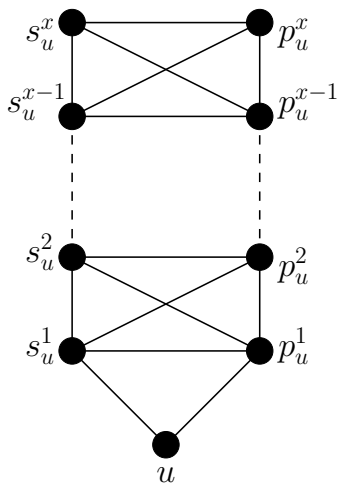
Let $x = \lfloor \frac{k}{2} \rfloor$. With each vertex u of G_F^c, we associate a gadget G_u made up of x consecutive *levels*. Each vertex from any level is joined to all vertices of the previous and next levels (if any). Besides, u is connected to the first level of G_u.



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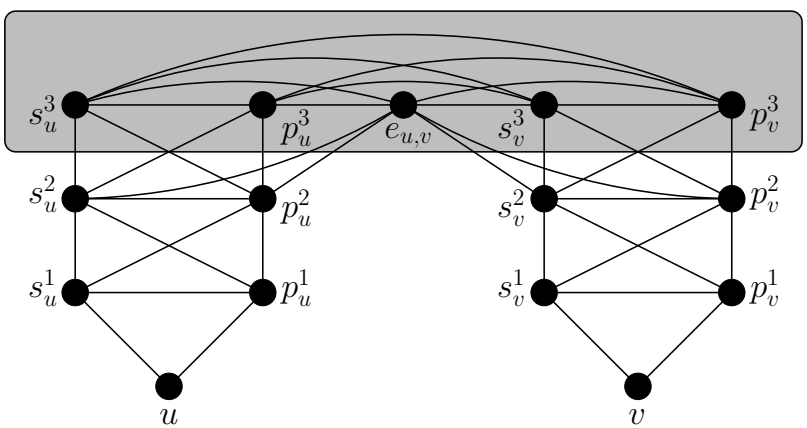
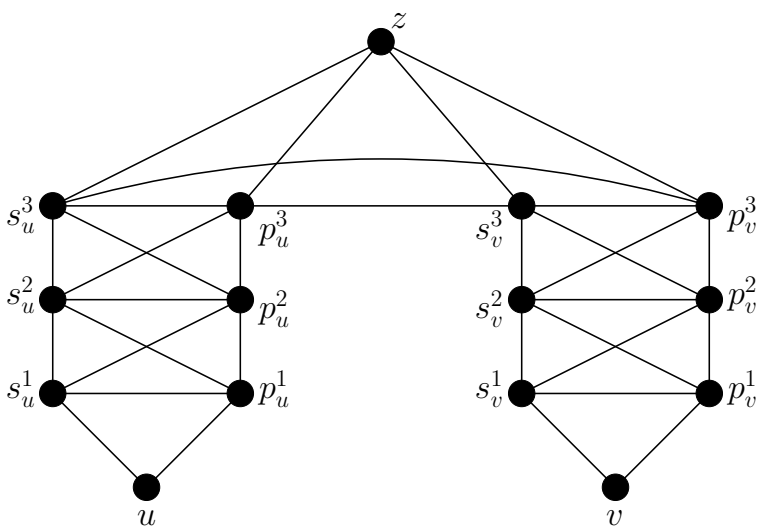
We now connect any two gadgets G_u and G_v. In the case where $\{u, v\}$ is not representative, we also add some *shortcut edges* between G_u and G_v. Typically, two vertices u' and v' from G_u and G_v are at distance at most k, unless $u' = u$, $v' = v$ and $\{u, v\}$ is representative.





Complexity of finding a k-weak orientation

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Deciding whether G admits a k-weak orientation is NP-complete for every $k \geq 2$.

Consider the following partial orientation of G_F. For each gadget G_u, orient the edges in such a way that the left column goes upwards, and the right one goes downwards. With this orientation, every vertex from G_u can "access" any level of G_u.

Now define an arbitrary ordering $(u_1, ..., u_{|V(G_F^c)|})$ over the vertices of G_F^c. Consider any two gadgets G_{v_i} and G_{v_j} with $i < j$. Orient the edges incident with the x^{th} levels so that the top-left vertex of G_{u_i} can access the top-right vertex of G_{u_j}, and vice-versa. Orient the shortcut edges similarly (if any).

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Now define an arbitrary ordering $(u_1, ..., u_{|V(G_F^c)|})$ over the vertices of G_F^c. Consider any two gadgets G_{v_i} and G_{v_j} with $i < j$. Orient the edges incident with the x^{th} levels so that the top-left vertex of G_{u_i} can access the top-right vertex of G_{u_j}, and vice-versa. Orient the shortcut edges similarly (if any).

Note that there is a path with length at most k starting from any vertex of G_{u_i}, reaching the top-left vertex, exiting G_{u_i}, reaching the top-right vertex of G_{u_j}, and going downwards within G_{u_j} when $i < j$. A similar directed path from G_{u_j} to G_{u_i} also exists in the other direction.

Complexity of finding a k-weak orientation

Thm. *B., Duvignau, Kirgizov (2013+)*

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Finally, if $\{u_i, u_j\}$ is not representative, then we have a directed path with length k passing by the shortcut edges between G_{u_i} and G_{u_j}. ■

Part 1: Oriented distances and diameters

Part 2: 2-weak orientations and oriented chromatic number

Part 3: Complexity of finding a k-weak orientation

Part 4: Open question

Open question

Is it possible to use our reduction scheme to prove that deciding whether a graph admits a k-strong orientation is NP-complete for every $k \geq 2$?

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Thank you for your attention.