Strong edge-coloring of $(3, \Delta)$-bipartite graphs

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Strong edge-coloring

G: undirected simple graph
c: edge-coloring of G

> ### Definition: *strong edge-coloring*
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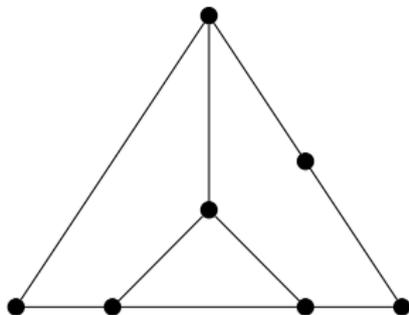> We call c *strong* if every two edges at distance at most 2 in G are assigned distinct colors by c.

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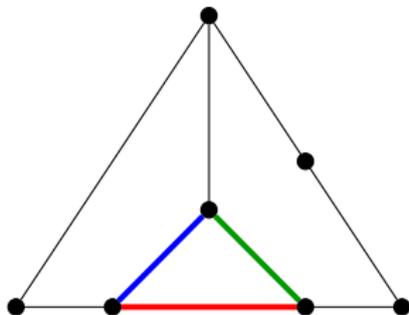> We call *c strong* if every two edges at distance at most 2 in *G* are assigned distinct colors by *c*.

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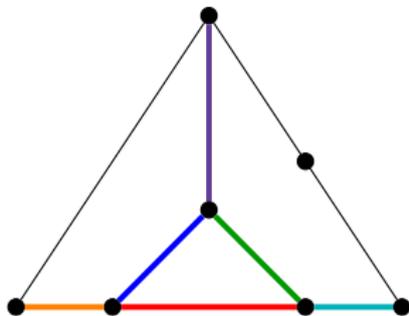> We call *c strong* if every two edges at distance at most 2 in *G* are assigned distinct colors by *c*.

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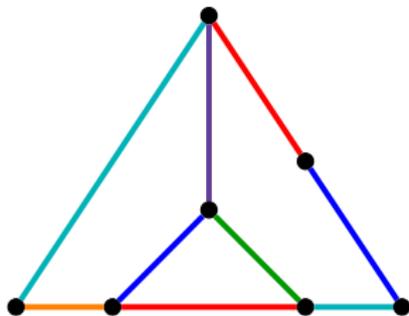> We call *c strong* if every two edges at distance at most 2 in *G* are assigned distinct colors by *c*.

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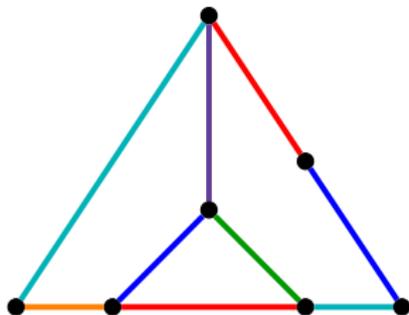> We call *c strong* if every two edges at distance at most 2 in *G* are assigned distinct colors by *c*.

Equivalently:
- edge-partition giving *induced* matchings
- proper vertex-coloring of $L(G)^2$

Strong chromatic index

Δ: maximum degree of an explicit graph

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> The least number of colors in a strong edge-coloring of G is the *strong chromatic index* of G, denoted $\chi'_s(G)$.

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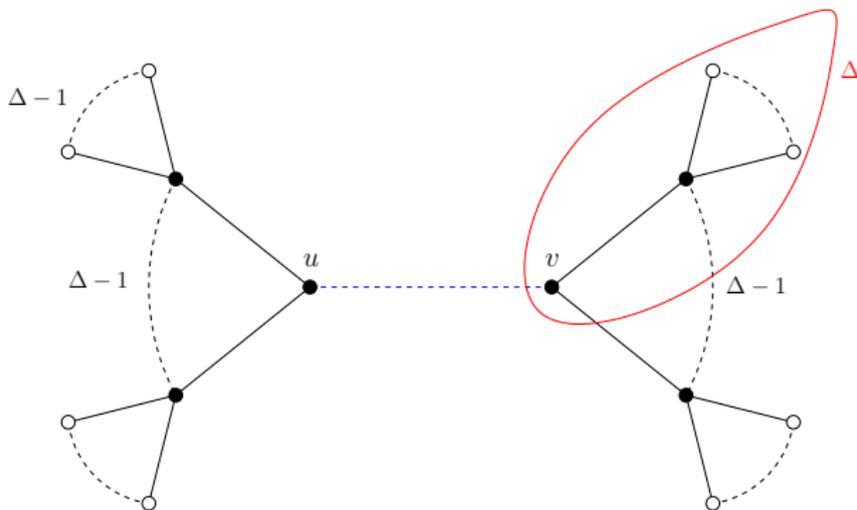
Brooks-like argument: $\chi'_s(G) \leq 2\Delta^2 - 2\Delta + 1$ ($\approx 2\Delta^2$)

On the Brooks-like upper bound on χ'_s

optimality of $2\Delta^2$?

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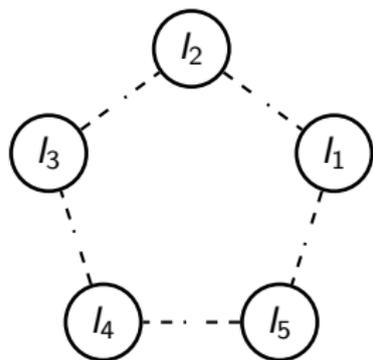
What would be a "worst graph"? C_5^Δ:

- every I_j is an independent set,

- two "adjacent" I_j's are complete to each other,

- if $\Delta = 2k$, then $|I_j| = k$,

- if $\Delta = 2k + 1$, then $|I_1| = |I_2| = |I_3| = k$,

  and $|I_4| = |I_5| = k + 1$.

Erdős and Nešetřil's conjecture

Conjecture [Erdős, Nešetřil – 1985]

We have $\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2 \text{ for } \Delta \text{ even, and} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1) \text{ otherwise.} \end{cases}$

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- reached for C_5^Δ's only [Chung, Gyárfás, Tuza, Trotter – 1990]

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- verified for $\Delta = 3$ [Andersen – 1992]

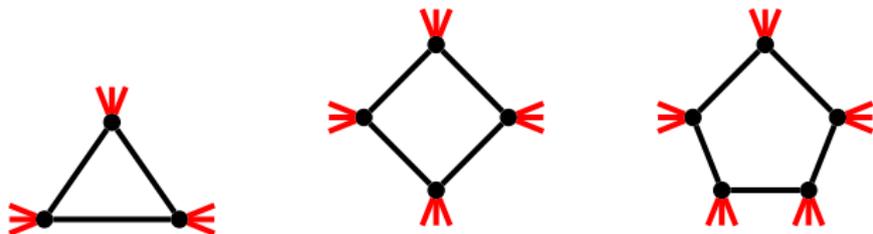# Erdős and Nešetřil's conjecture

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- reached for C_5^Δ's only [Chung, Gyárfás, Tuza, Trotter – 1990]
- verified for $\Delta = 3$ [Andersen – 1992]
- for $\Delta = 4$, we know $\chi'_s(G) \leq 22$ [Cranston – 2006]
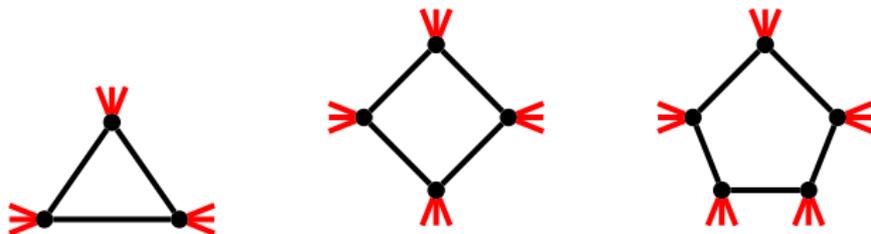
Beyond Erdős and Nešetřil's construction

Less dependencies for graphs with no small cycles

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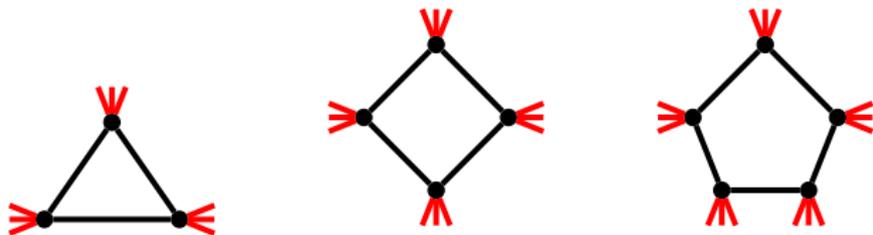Less dependencies for graphs with no small cycles

Theorem [Mahdian – 2000]

If G is C_4-free, then $\chi'_s(G) \leq (2 + o(1))\frac{\Delta^2}{\ln \Delta}$.

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What for C_3- and C_5-free graphs?

What for bipartite graphs?

Bipartite graphs are C_3 and C_5-free...

> **Conjecture [Faudree, Gyárfás, Schelp, Tuza – 1990]**
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> If G is bipartite, then $\chi'_s(G) \leq \Delta^2$.

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$G = (A, B, E)$: bipartite graph with bipartition A and B
(Δ_A, Δ_B)-bipartite graph: A and B have maximum degree Δ_A and Δ_B, resp.

> **Conjecture [Brualdi, Quinn Massey – 1993]**
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> If G is (Δ_A, Δ_B)-bipartite, then $\chi'_s(G) \leq \Delta_A \Delta_B$.

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Verified when:

- $\Delta_A = \Delta_B = 3$ [Steger and Yu – 1993]
- $\Delta_A = 2$ [Nakprasit – 2008]

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We confirm the conjecture when $\Delta_A = 3$ and $\Delta_B \geq 4$

> **Theorem [B., Lagoutte, Valicov – 2014+]**
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> If G is $(3, \Delta_B)$-bipartite, then $\chi_s'(G) \leq 4\Delta_B$.
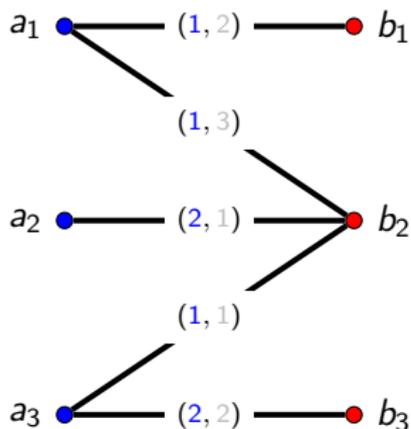
$(3, \Delta_B)$-bipartite graphs – a proof

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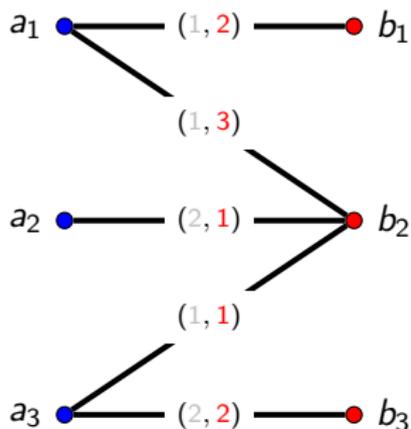
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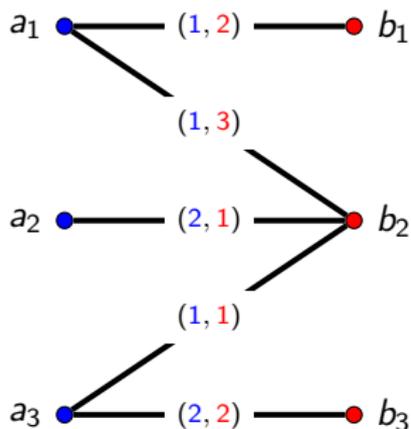
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Mixing c_A and c_B to get c

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As c_B, just consider a proper Δ_B-incidence coloring
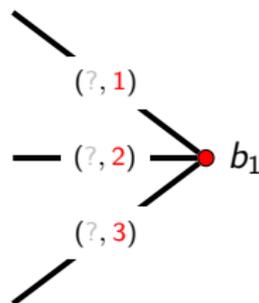
(adjacent incidences are of the form (b_1, e) and (b_1, f))

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Coloring procedure:
    Step 1: color the edges incident to Type 1 vertices
    Step 2: color the paired edges incident to Type 2
    Step 3: color the edges incident to Type 3 vertices
    Step 4: color the lonely edges incident to Type 2 vertices

Step 1: color the edges incident to Type 1 vertices

Just assign the colors among $\{1, 2, 3\}$ greedily

Lemma

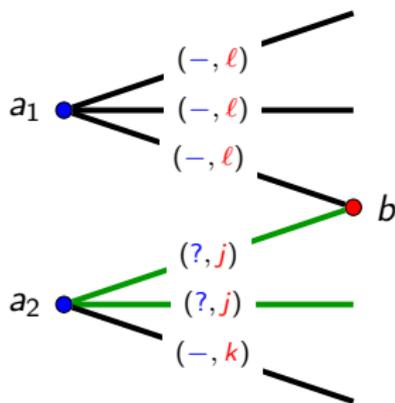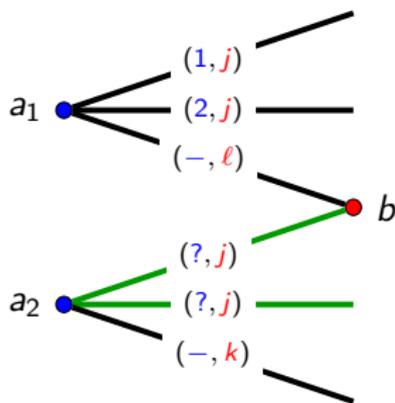There is at least one available color for every edge to color.

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Proof. There may be a forbidden color $(-, j)$ adjacent to the bottom-most edge
– this is the only one since c_B is proper and "maximum"
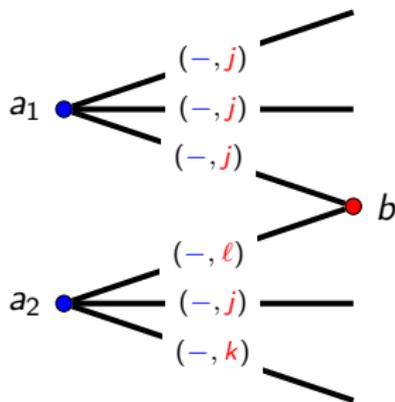

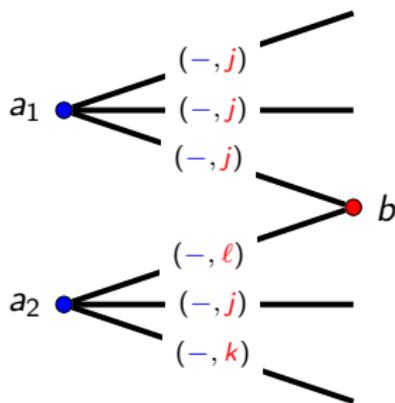if a_1 is Type 1, then the colors are different

Step 2: color the paired edges incident to Type 2 vertices

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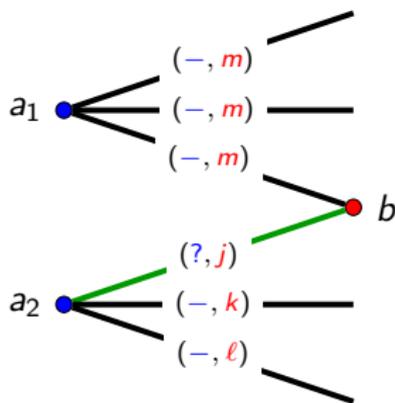There is at least one available color for every edge to color.

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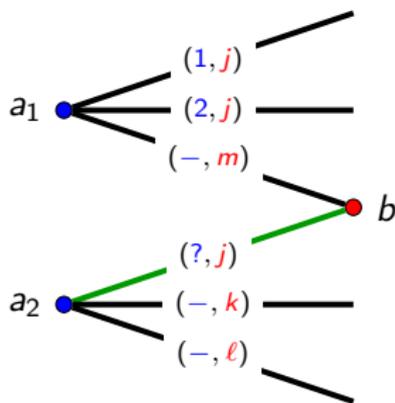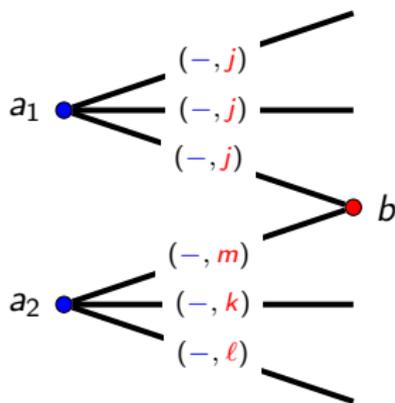

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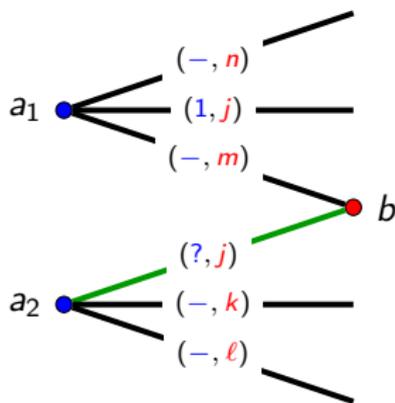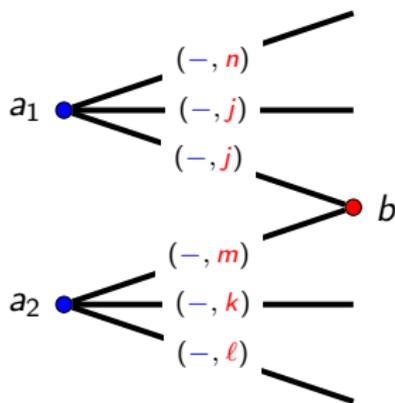

if a_1 is Type 2...
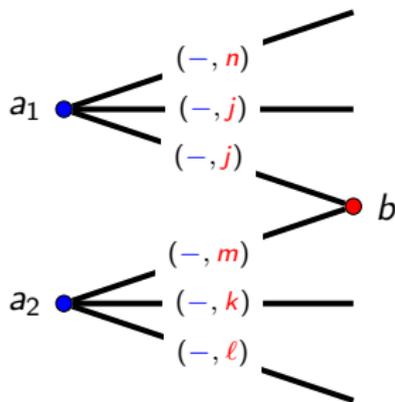
Step 3: color the edges incident to Type 3 vertices

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$$(-, n)$$
$$a_1 \quad (1, j)$$
$$(-, m)$$
$$b$$
$$(?, j)$$
$$a_2 \quad (-, k)$$
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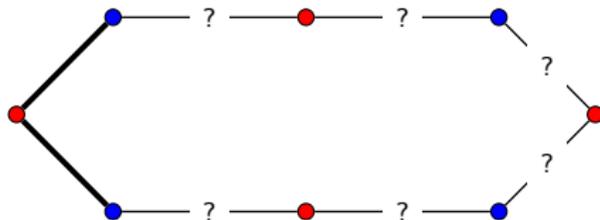# Step 4: color the lonely edges incident to Type 2 vertices

$\mathcal{C}_j$: (connected) subgraph induced by the j-lonely edges (*i.e.* with $c_B = j$)

Alternate cycle of $\mathcal{C}_j$: edges alternate between j-lonely and non-j-lonely

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Every cycle of $\mathcal{C}_j$ is alternate.

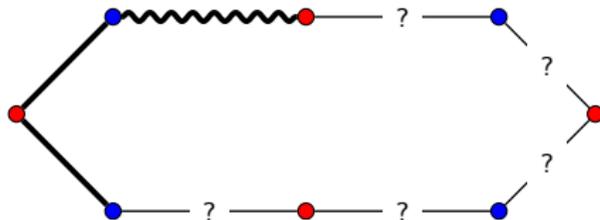# Step 4: color the lonely edges incident to Type 2 vertices

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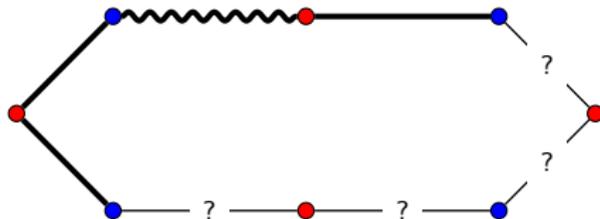# Step 4: color the lonely edges incident to Type 2 vertices

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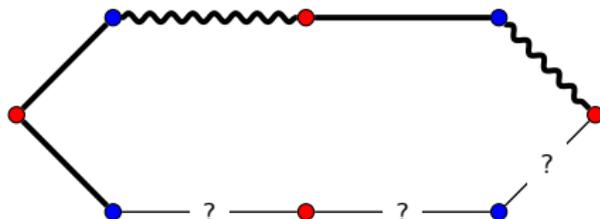

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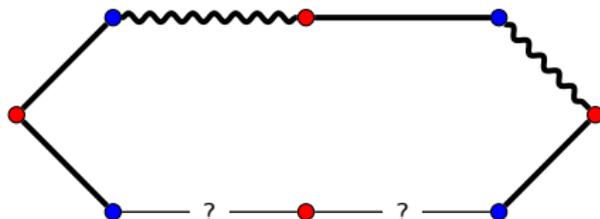# Step 4: color the lonely edges incident to Type 2 vertices

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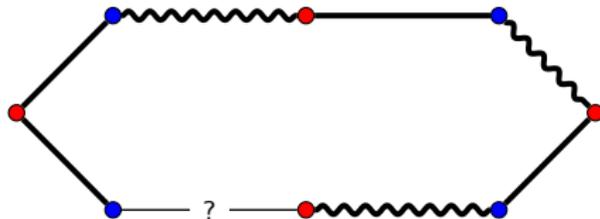# Step 4: color the lonely edges incident to Type 2 vertices

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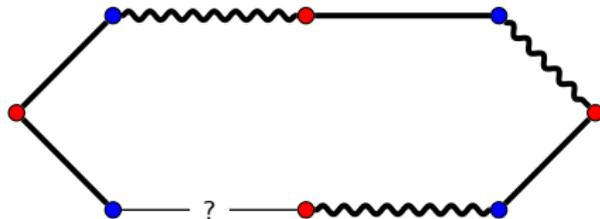# Step 4: color the lonely edges incident to Type 2 vertices

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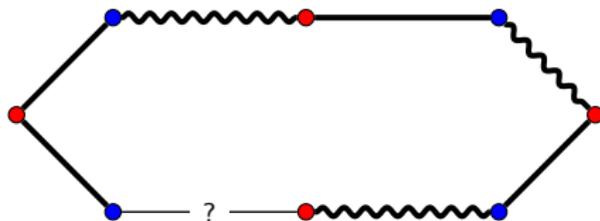
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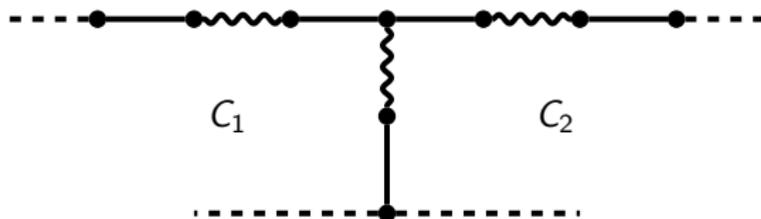
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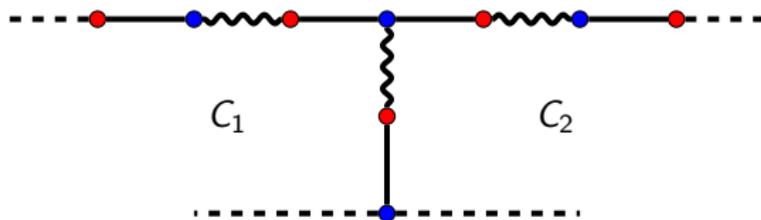
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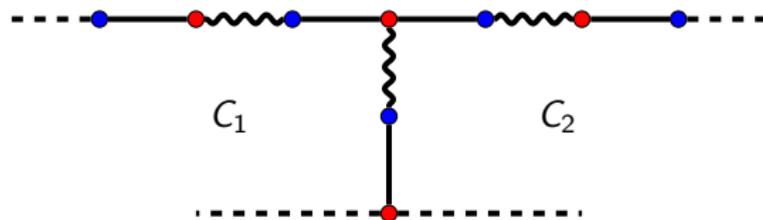

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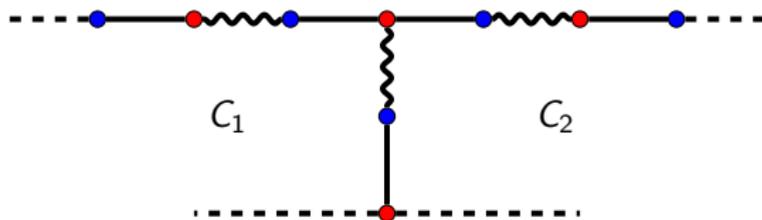

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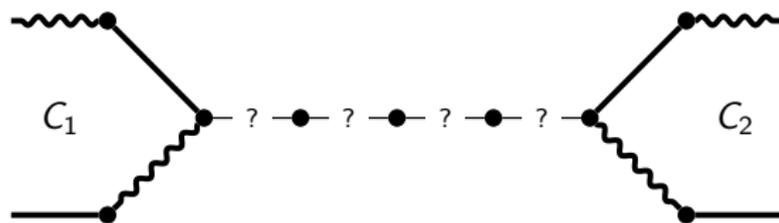Second end point of the intersecting path cannot be correct

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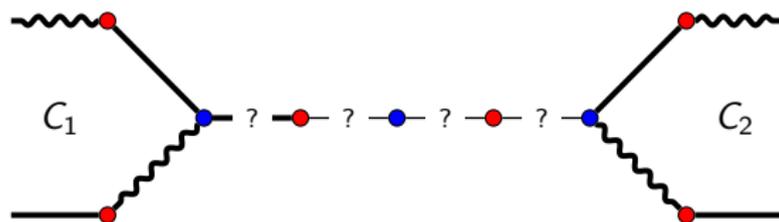**Proof.** Assume C_1 and C_2 are linked by a path $u...v$ where $u \in C_1$ and $v \in C_2$

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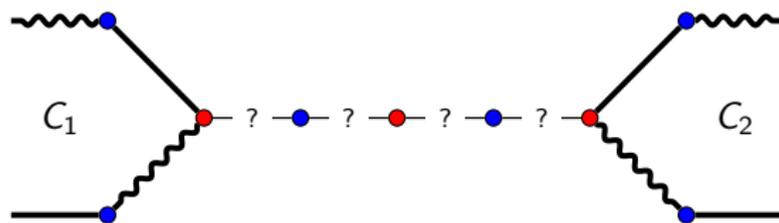
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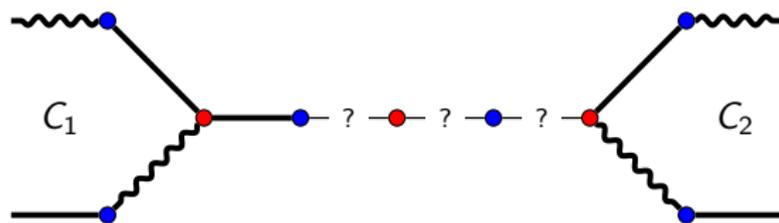

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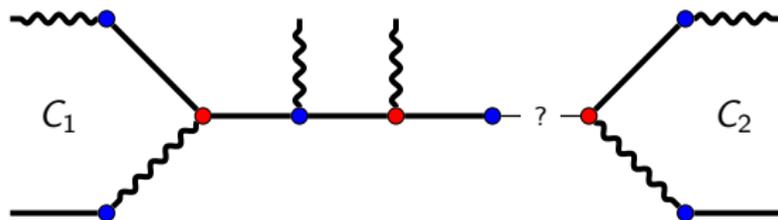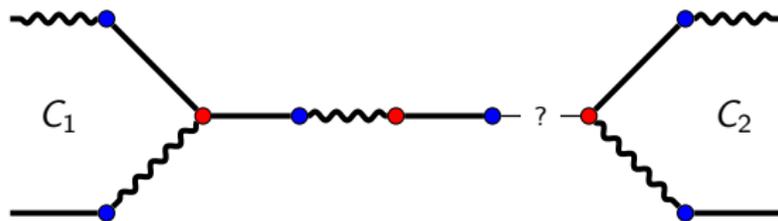**Proof.** Assume C_1 and C_2 are linked by a path $u...v$ where $u \in C_1$ and $v \in C_2$

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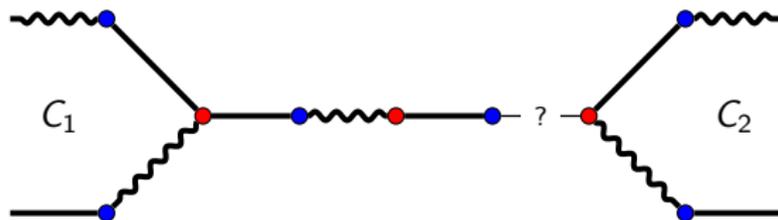

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Back to coloring Step 4

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  Phase 1: color the unique cycle C of $\mathcal{C}_j$
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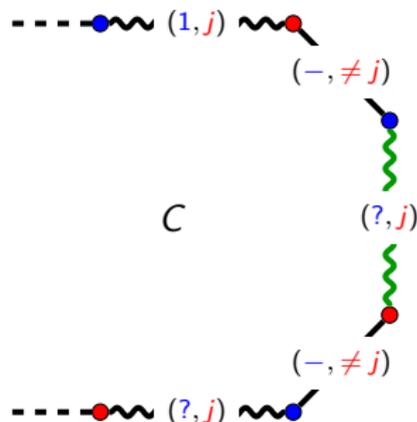
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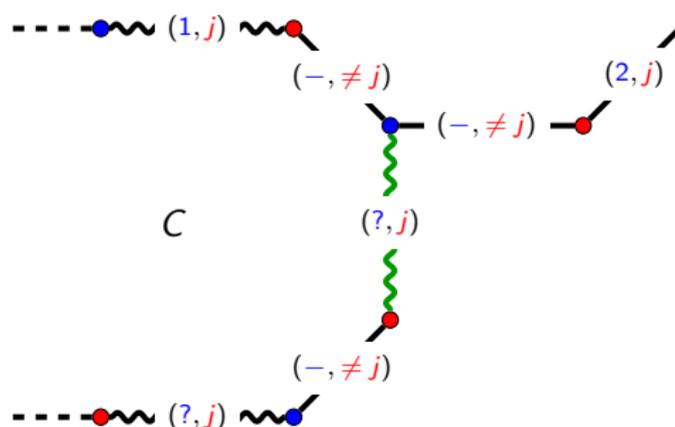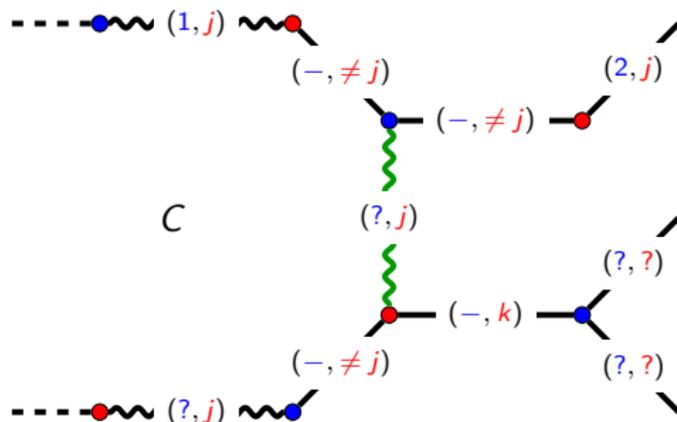
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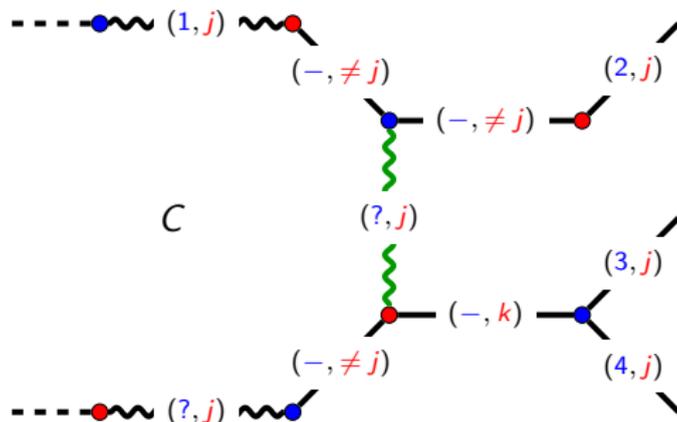Not yet colored

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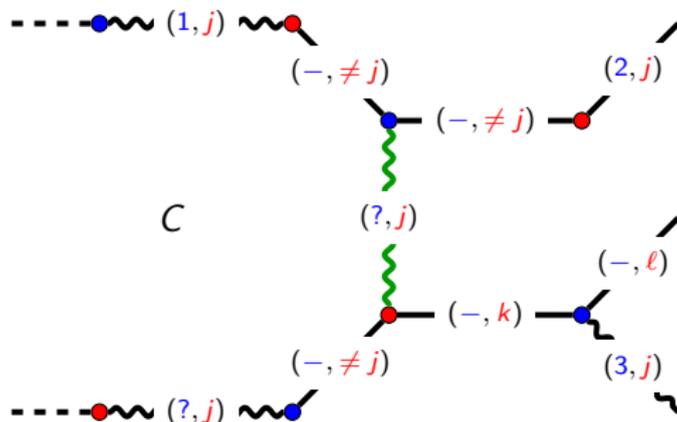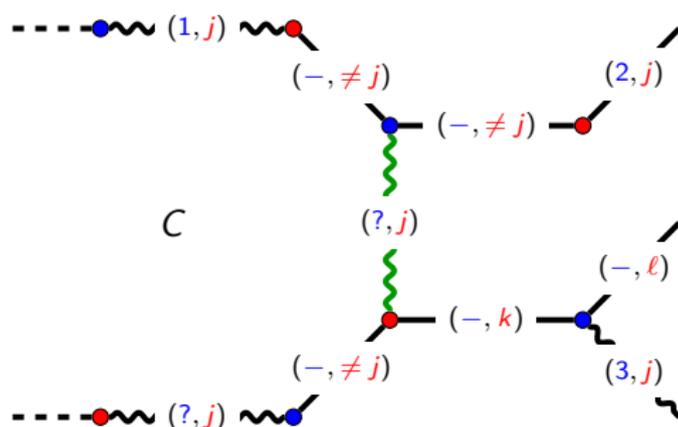   **Phase 1:** color the unique cycle C of $\mathcal{C}_j$

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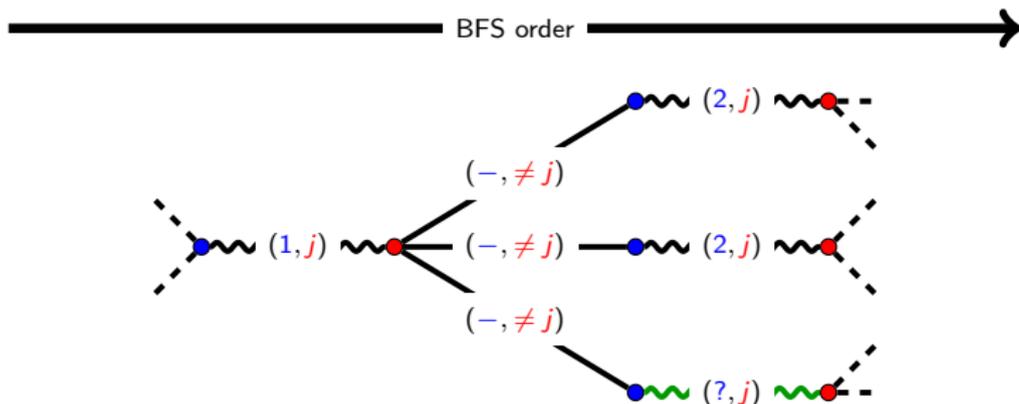
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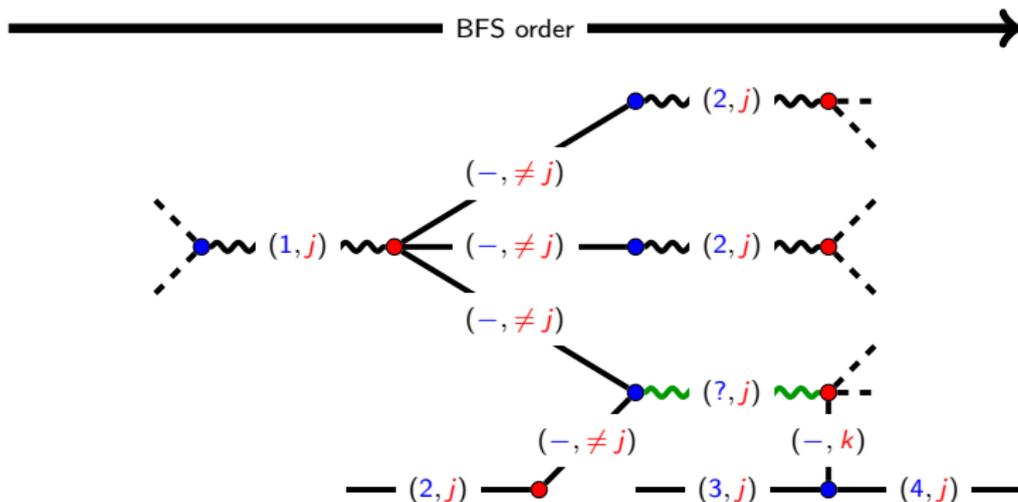

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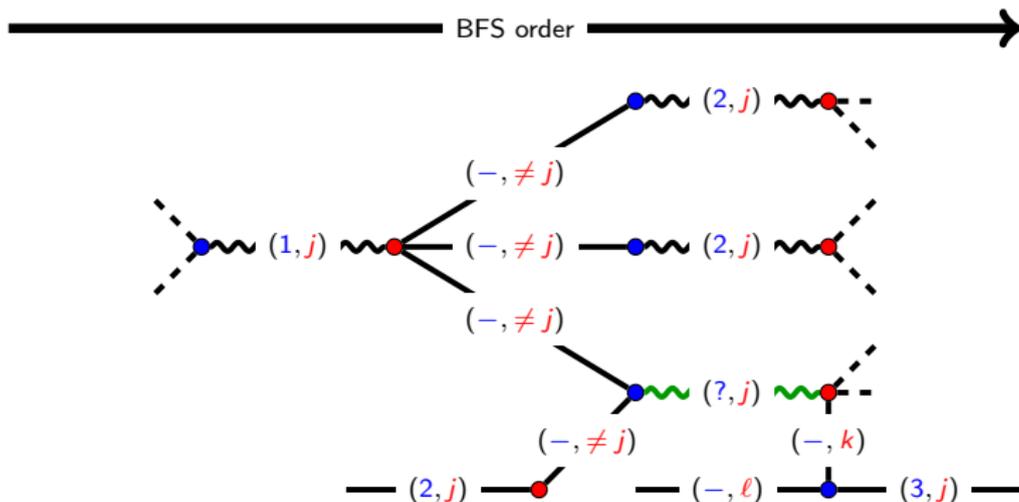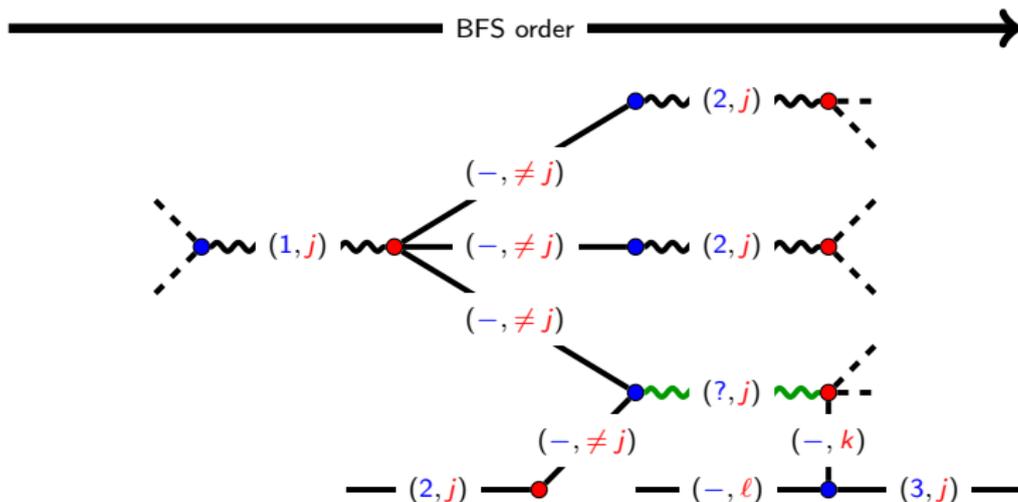

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Thank you for your attention.