

Sequential Metric Dimension (in trees)

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Introduction to the problem

Rules:

- Graph $G = (V, E)$;
- “Secret” vertex $t \in V$;
- Probing a vertex $v \Rightarrow \text{dist}_G(v, t)$.

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This minimum = Metric Dimension $\text{MD}(G)$ of G .

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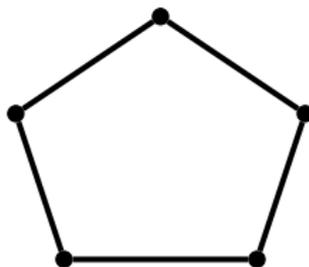


$$\text{MD}(P_n) = 1$$

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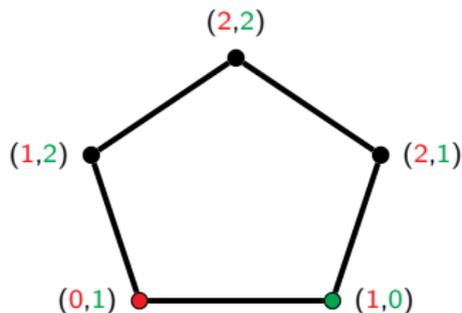
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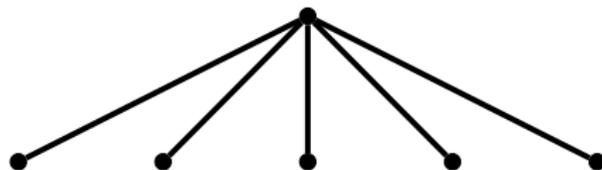


$$\text{MD}(C_n) = 2$$

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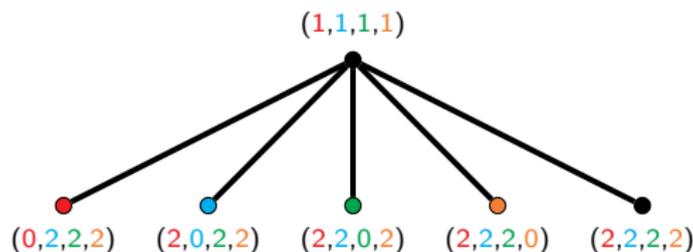


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$$\text{MD}(S_n) = n - 1$$

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Min. # of steps = Sequential Location Number $SL(G)$ of G .

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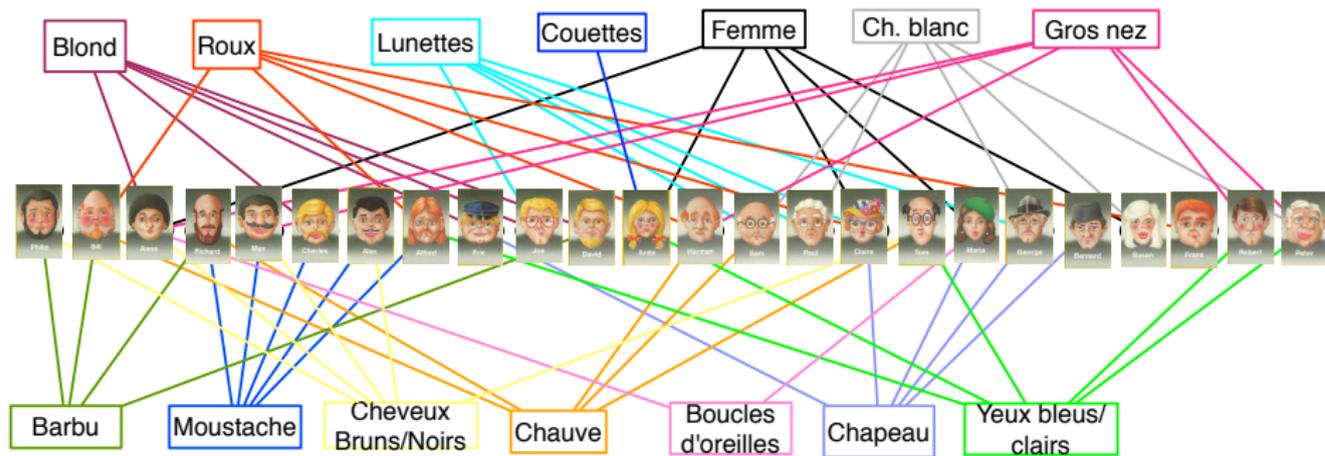
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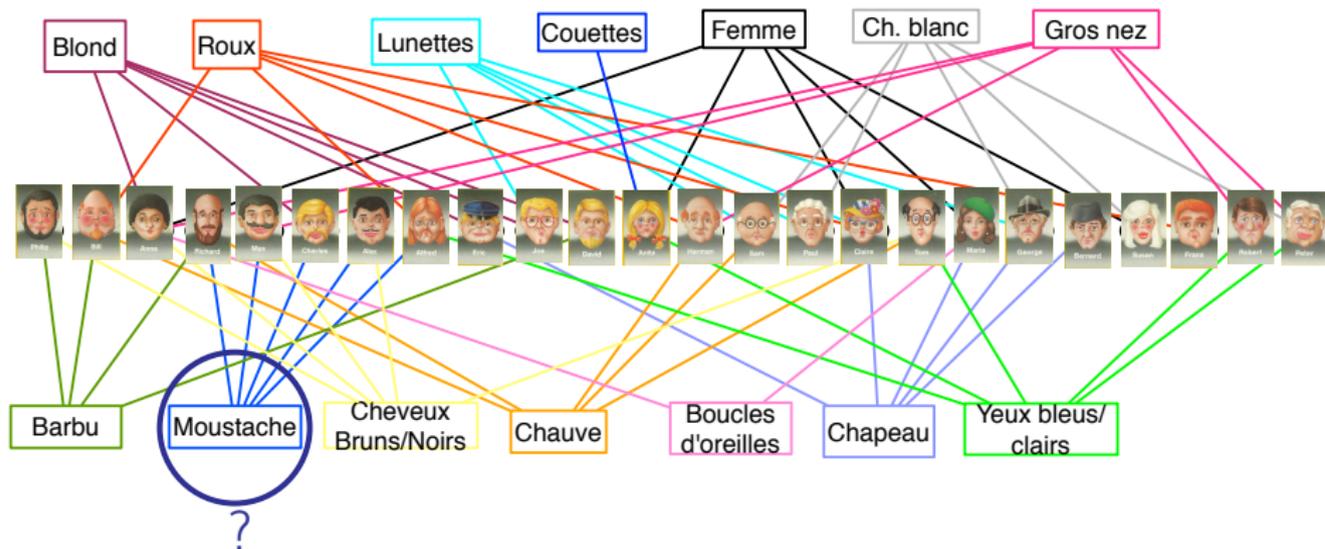
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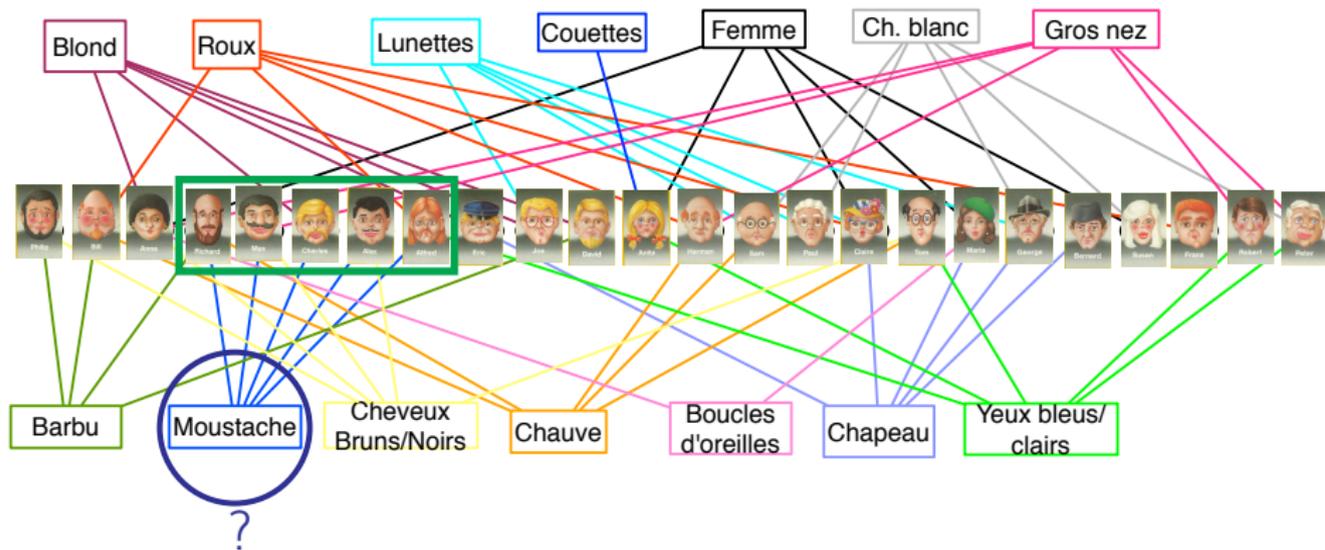
Sequential Locating Game and Guess Who?



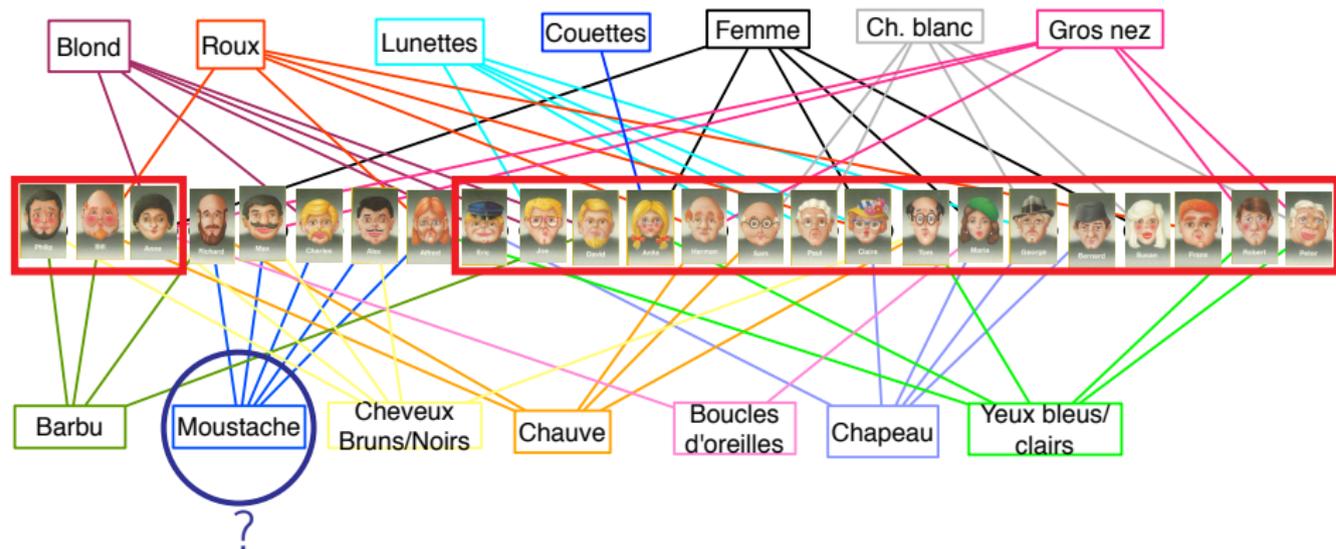
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Question

Given G, k, ℓ , is it possible to locate an immobile invisible target in G in at most ℓ steps, by probing at most k vertices each step?

Related:

- $\ell = 1$ (Metric Dimension);
- $k = 1$ (Sequential Locating Game);
- Moving target (Bosek et al., 2017).

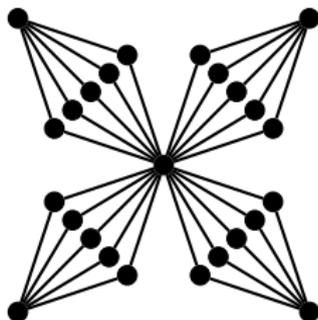
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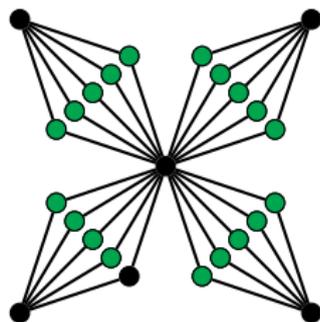
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All vertices in green probed



$\text{MD}(G) = 19$

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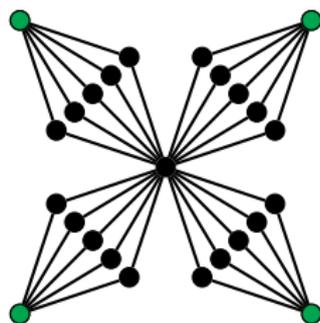
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Location in 2 steps; FIRST step:



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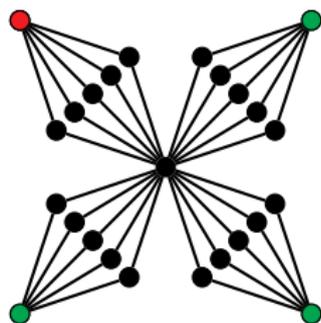
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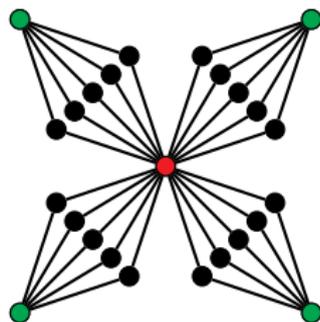
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Answer: $(2, 2, 2, 2)$

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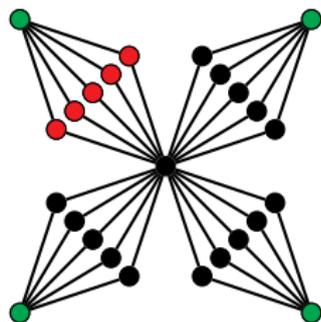
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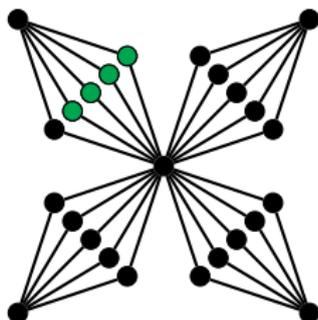
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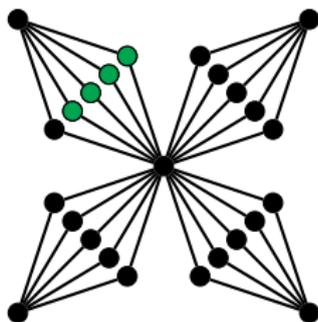
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$\Rightarrow \text{MD}(G) = 19$. And $\lceil \text{MD}(G)/4 \rceil = 5$, while 2 steps suffice.

Sequential Metric Dimension in trees

$\lambda_k(T)$: min. # of steps to locate t in T (probing at most k vertices each step).

Localisation Problem

Given a tree T , can we locate an **immobile invisible target** in at most ℓ steps, provided we can probe at most k vertices each step?

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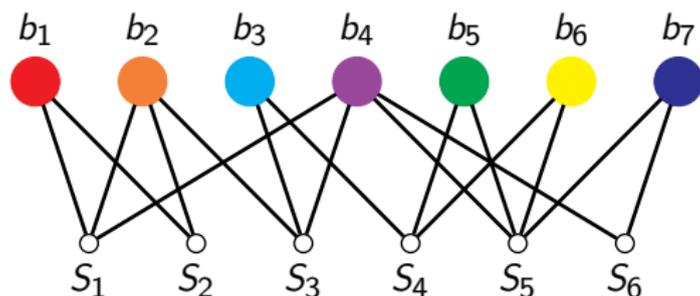
Tree case (k fixed, minimize ℓ):

- Making the appropriate first probing step is NP-complete ☹ ...
- ... but deciding how to probe optimally afterwards is polytime doable ☺ .
- \Rightarrow Polytime (+1)-approximation algorithm, yielding $\lambda_k(T)$ or $\lambda_k(T)+1$.

Theorem

Determining $\lambda_k(T)$ is NP-hard in trees T .

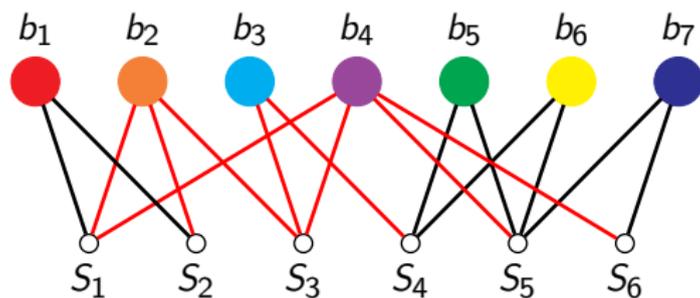
Proof (sketch). Reduction from Hitting Set (given a set $B := \{b_1, \dots, b_n\}$ and a set $\mathcal{S} := \{S_1, \dots, S_m\}$ of subsets of B , find a smallest subset of B hitting all S_i 's).



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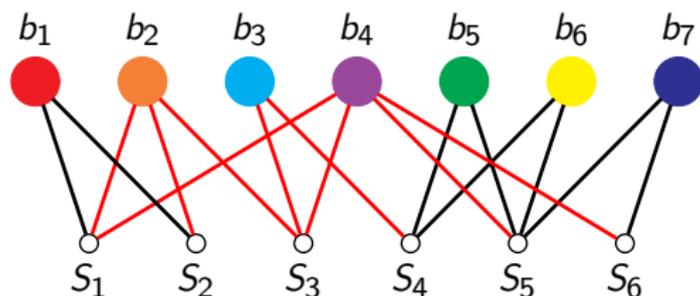
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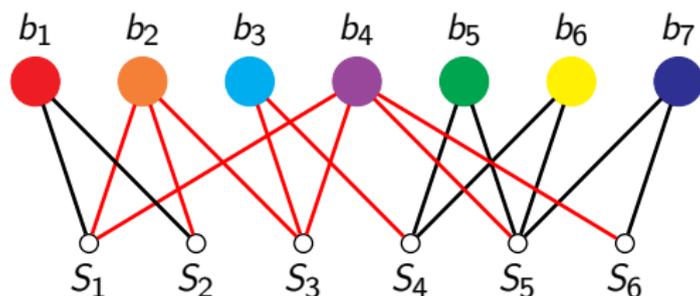
Main ideas:

- Have many **big stars** in the tree, so that the target has to hide in one such.
- Spend a few steps identifying the hosting big star, and then “peel” its leaves.

Theorem

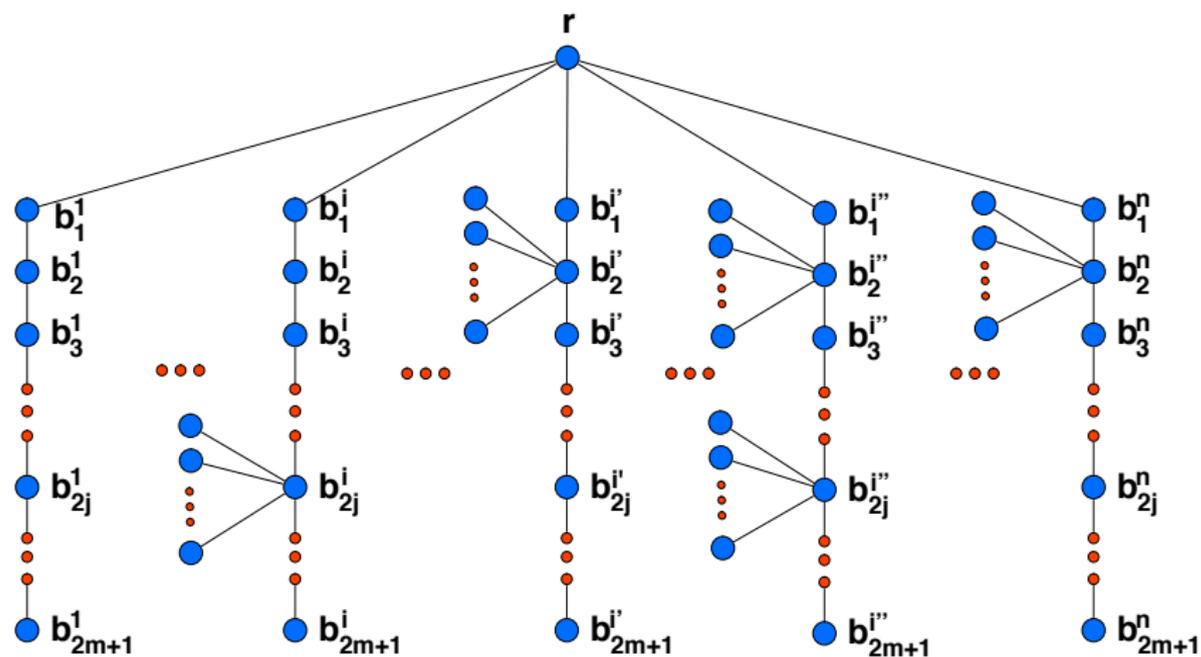
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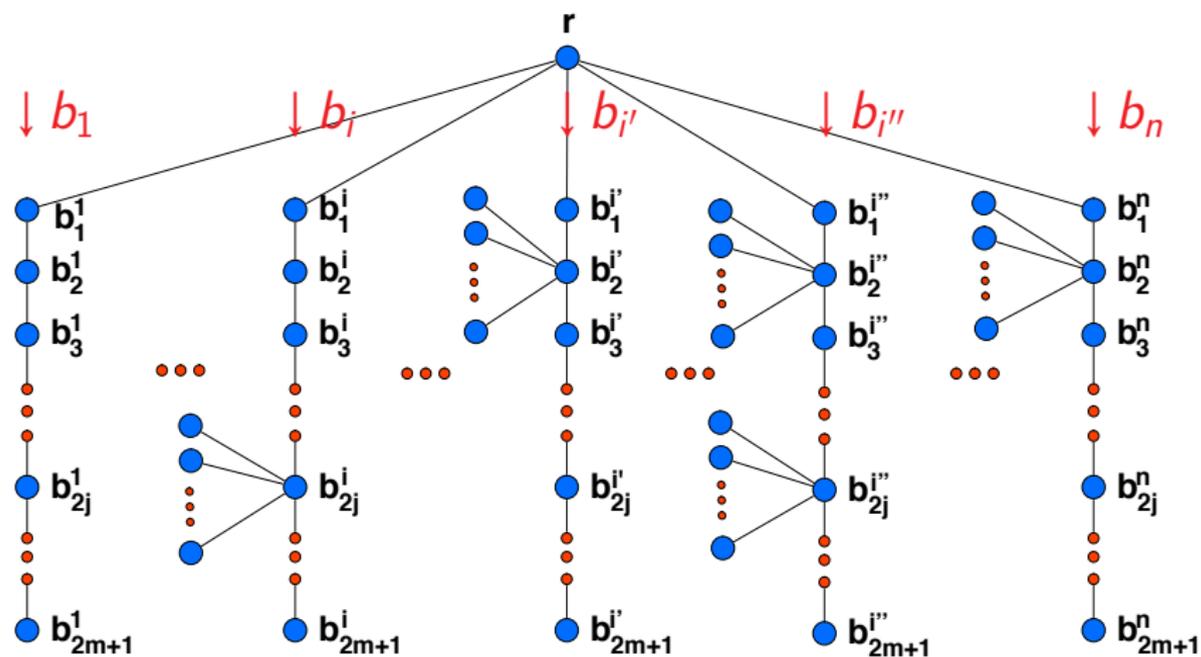
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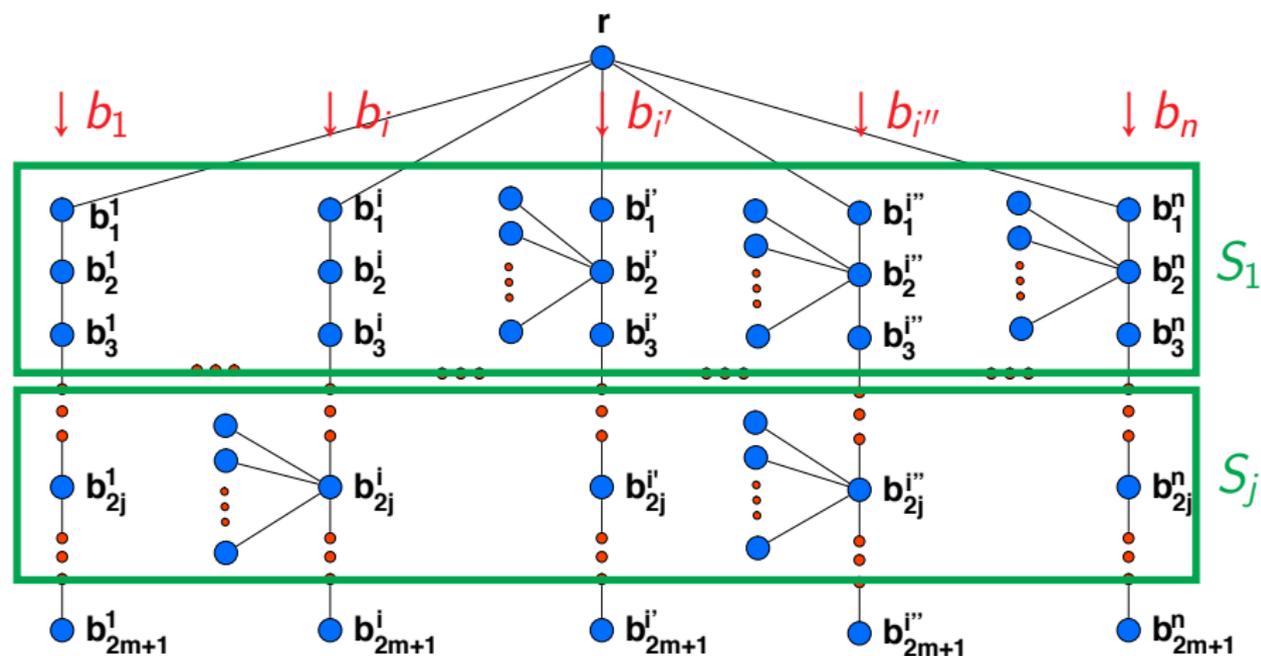


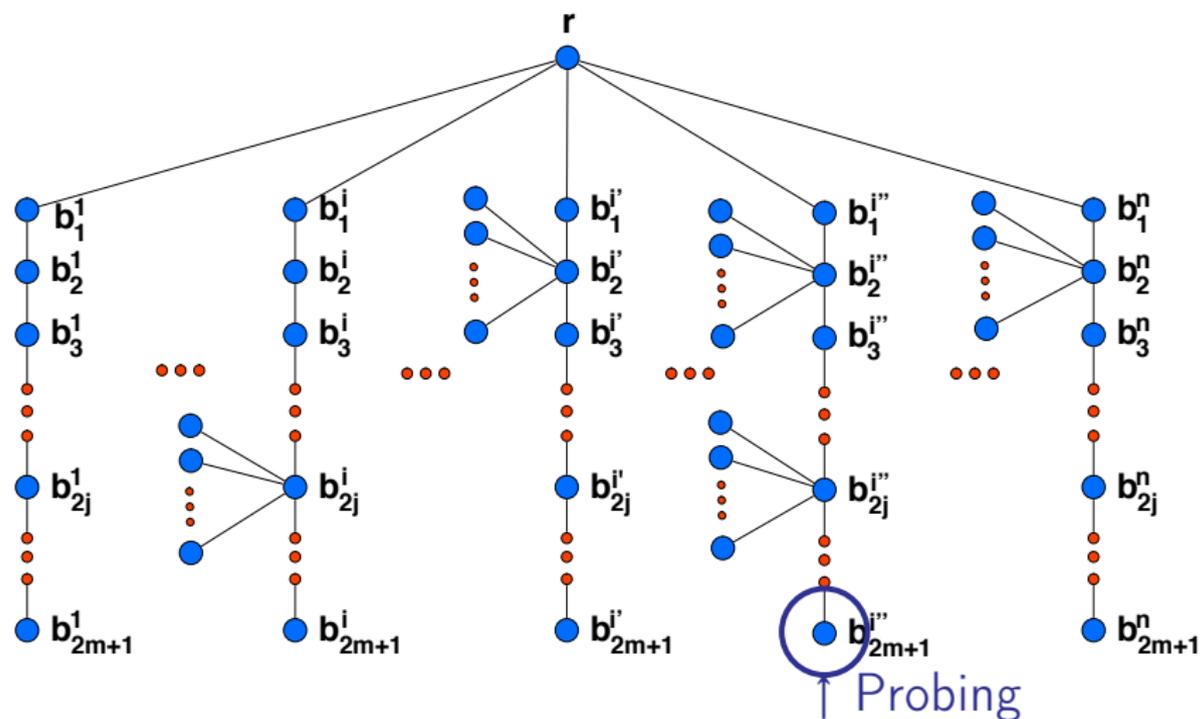
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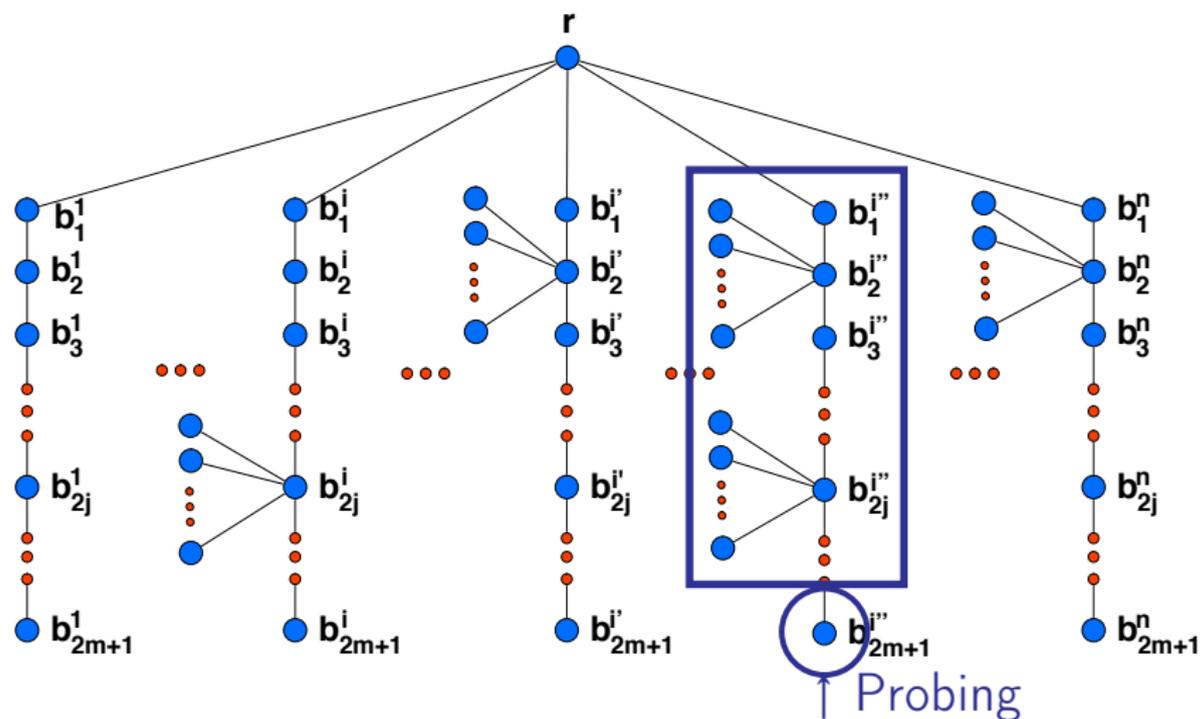
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 - Spend a few steps identifying the hosting big star, and then “peel” its leaves.
- ⇒ Identifying the big star early \Leftrightarrow Hitting set.

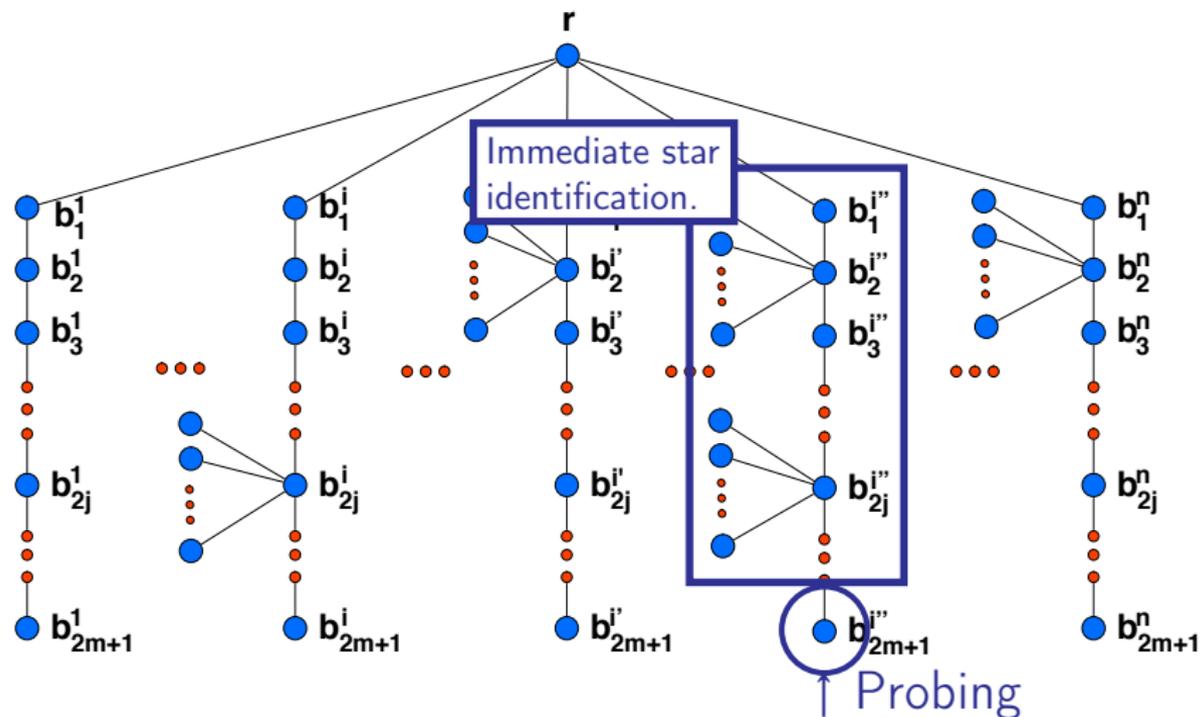


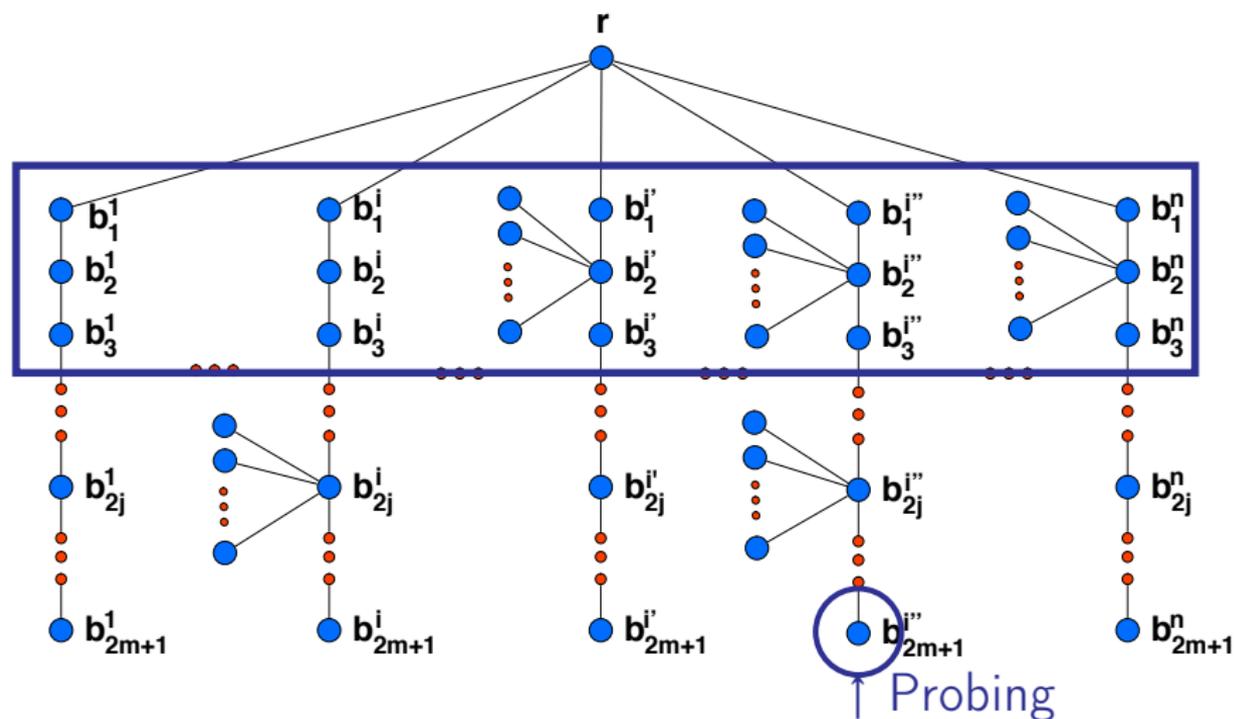


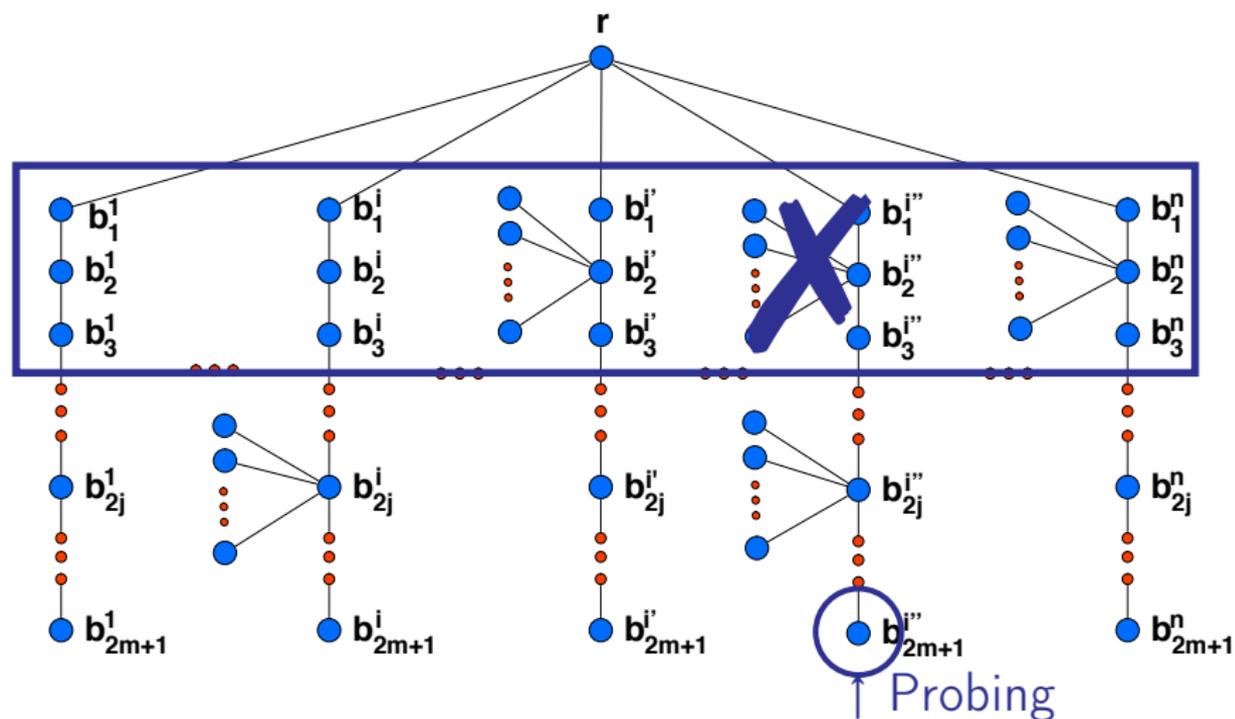


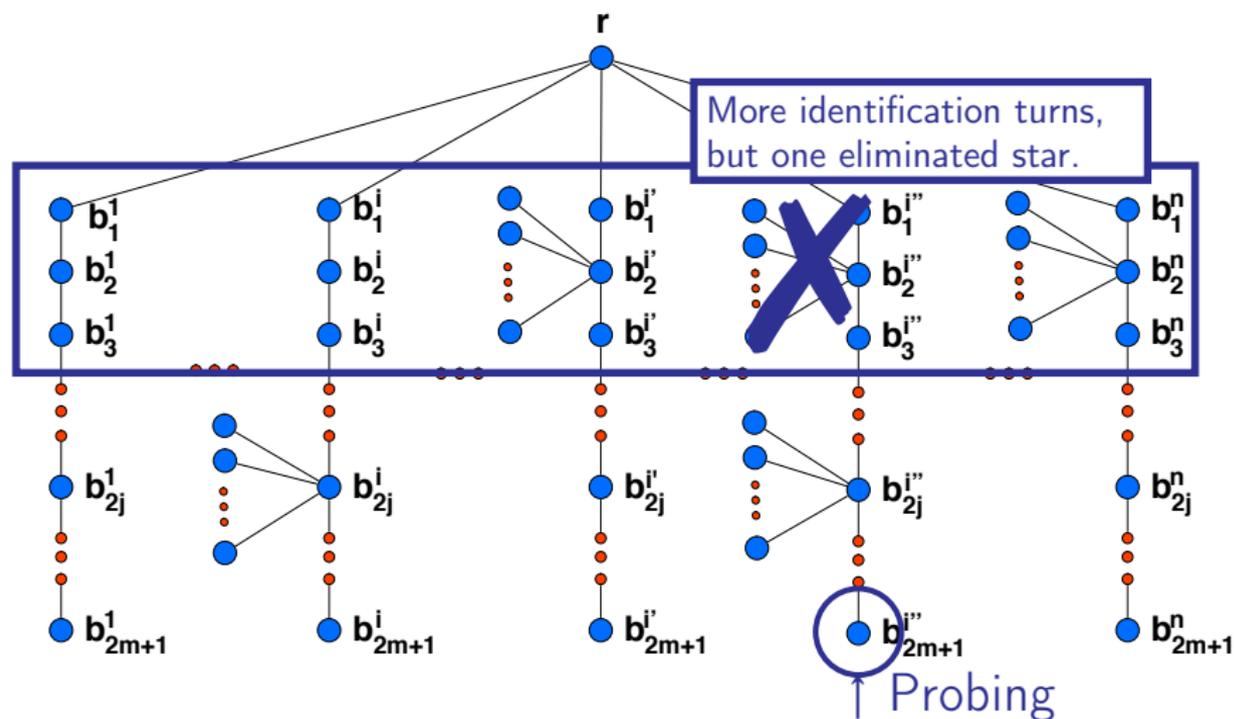


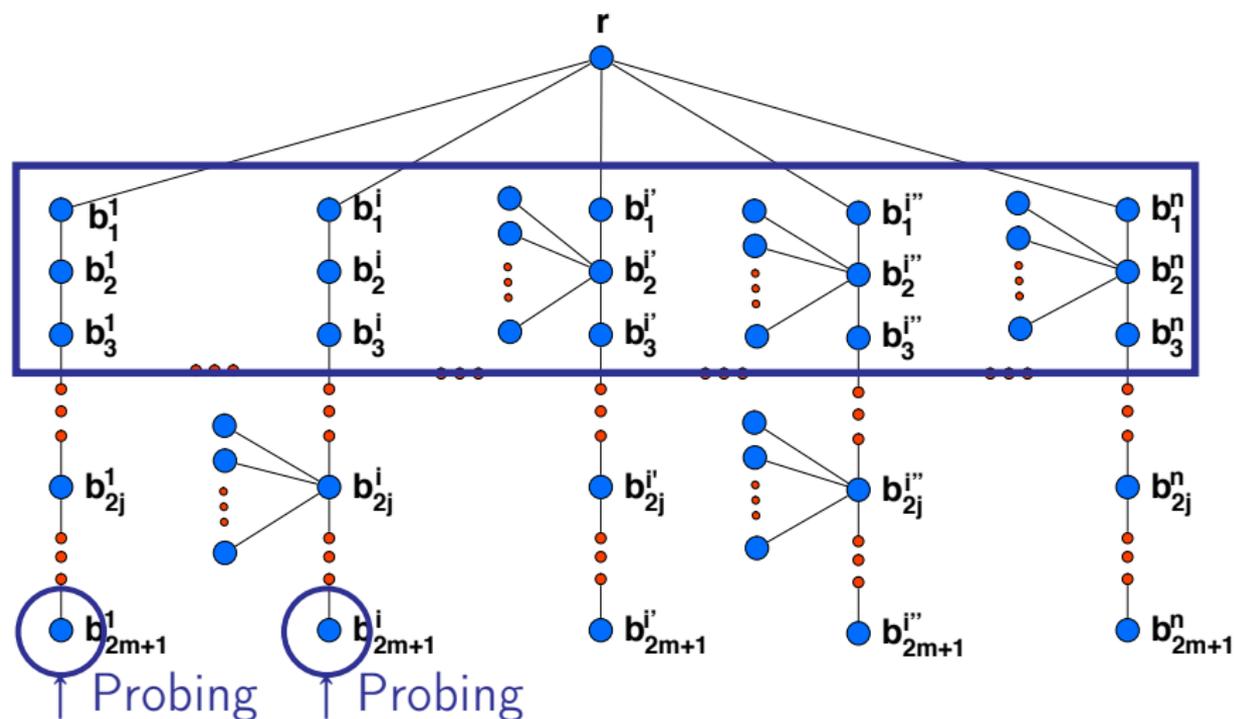


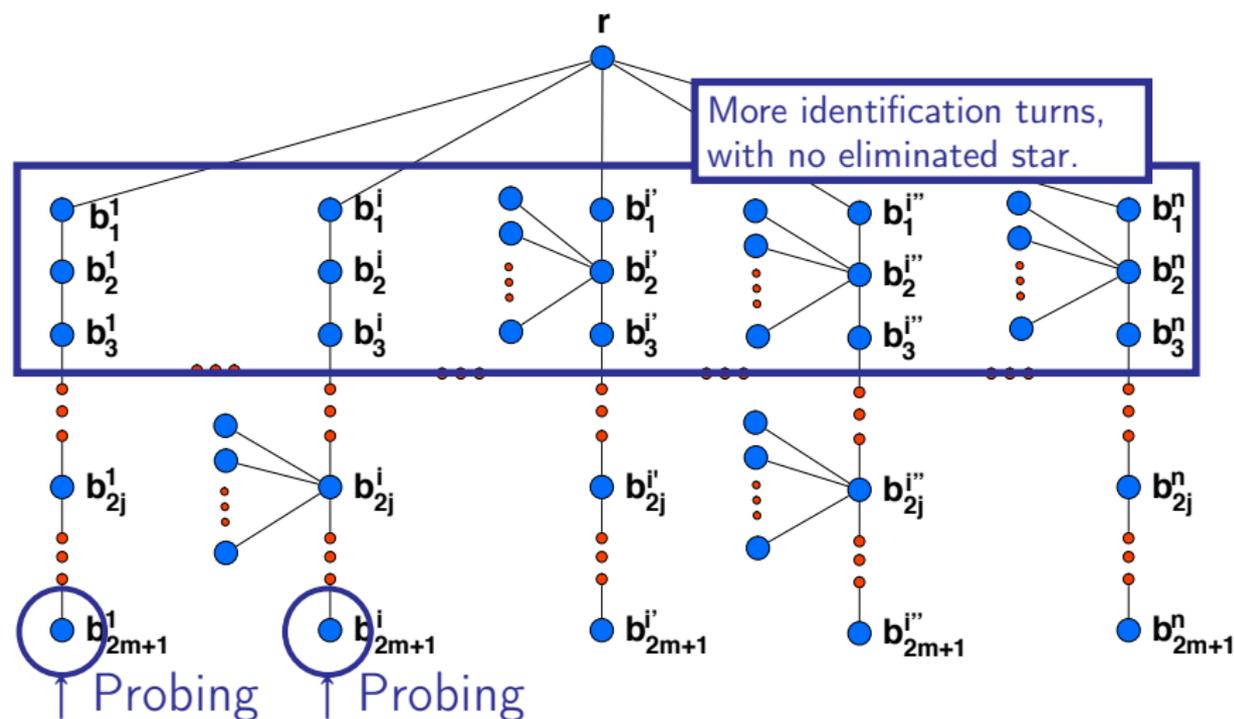






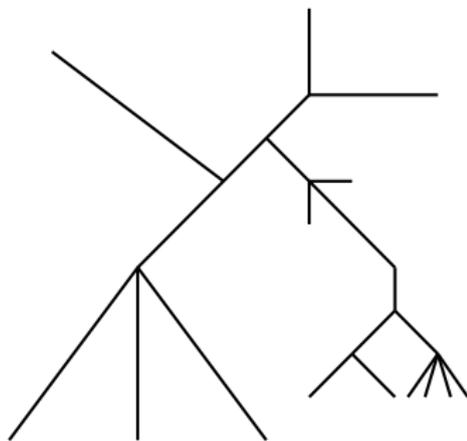




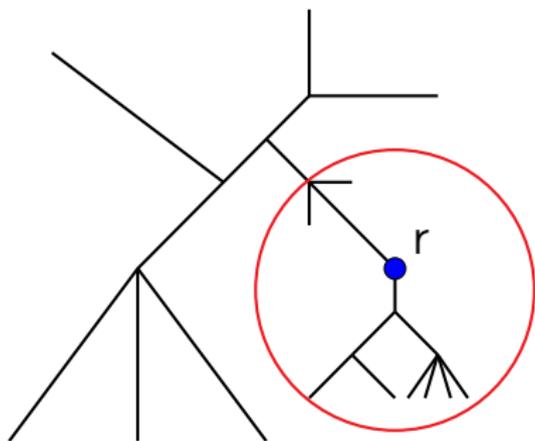


Want: First turn such that all S_i 's are hit... \Rightarrow Hitting set.

First step: Probe any one vertex r ...



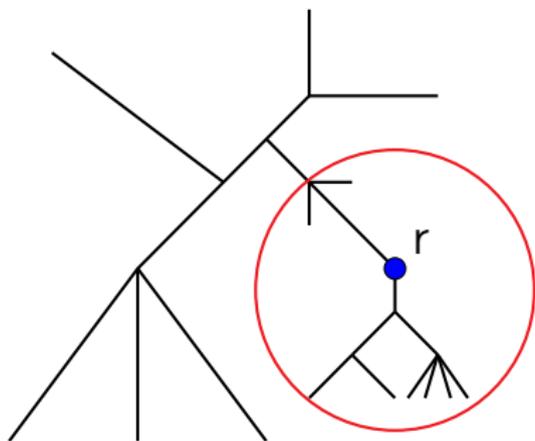
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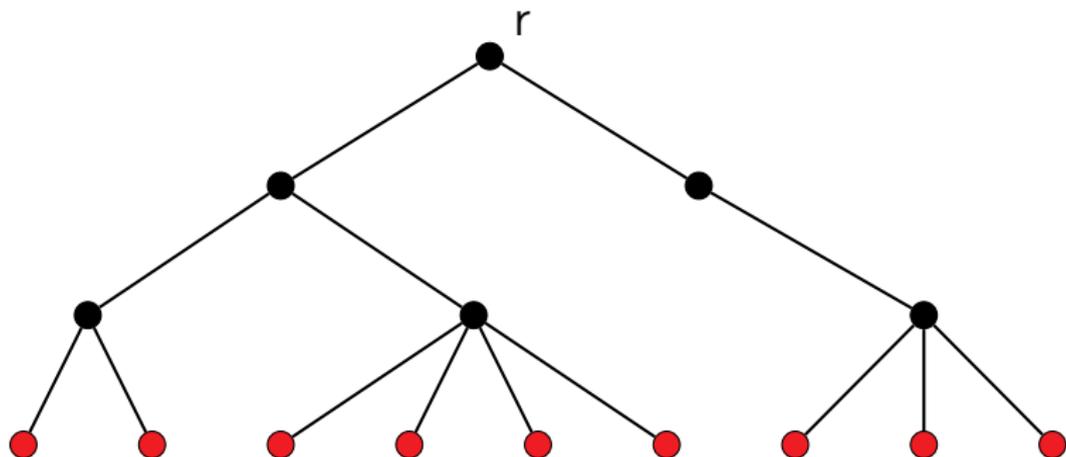
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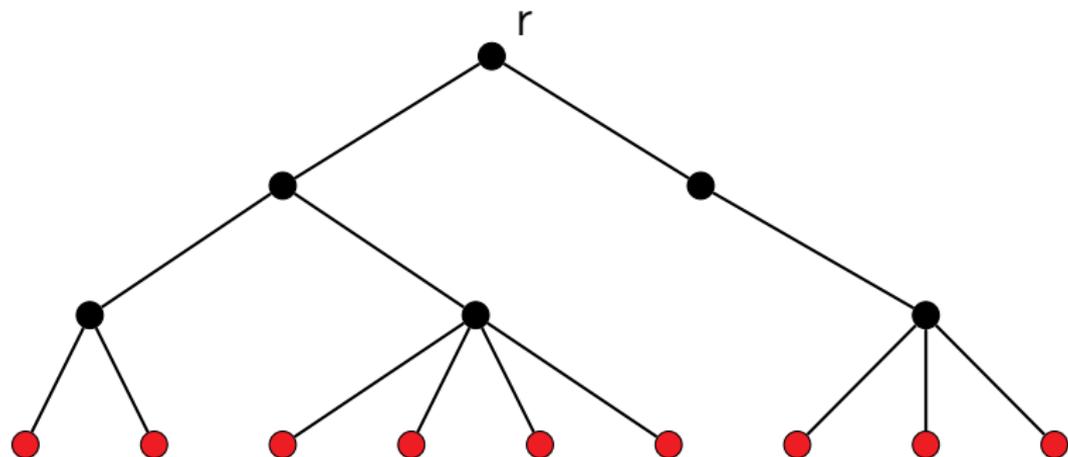
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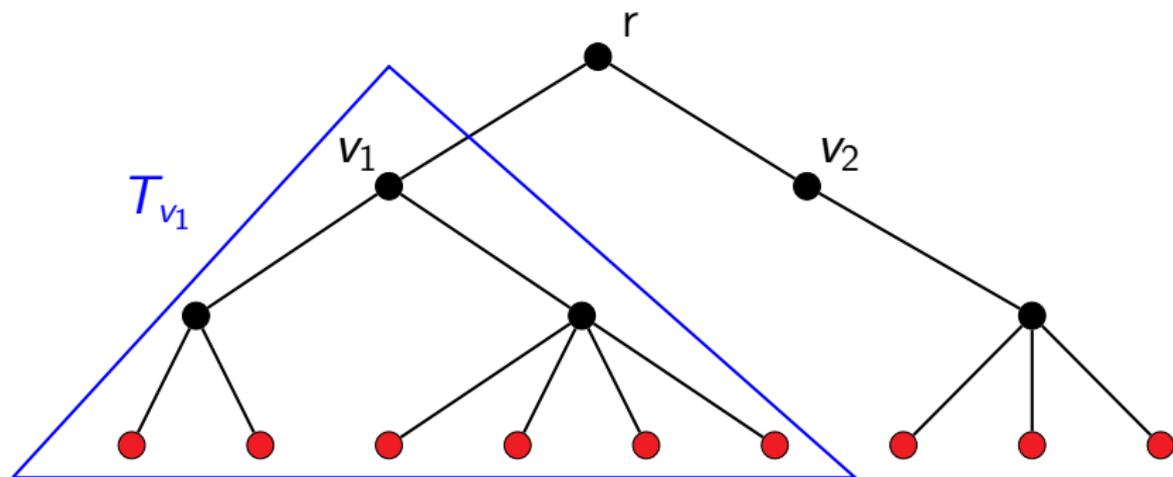
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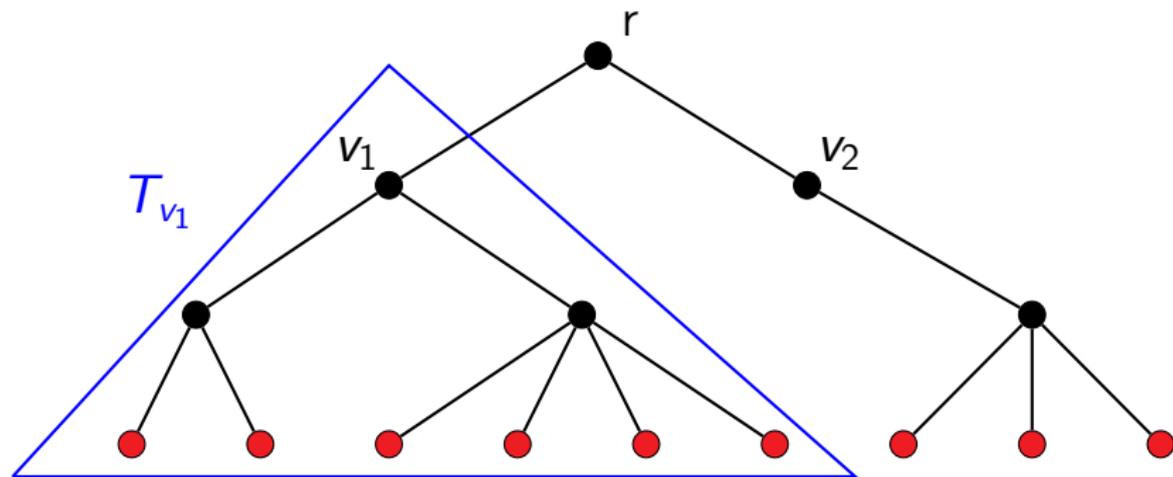


(because playing outside T' is pointless)

T_v : subtree rooted in v of T' rooted in r (v is a child of r).

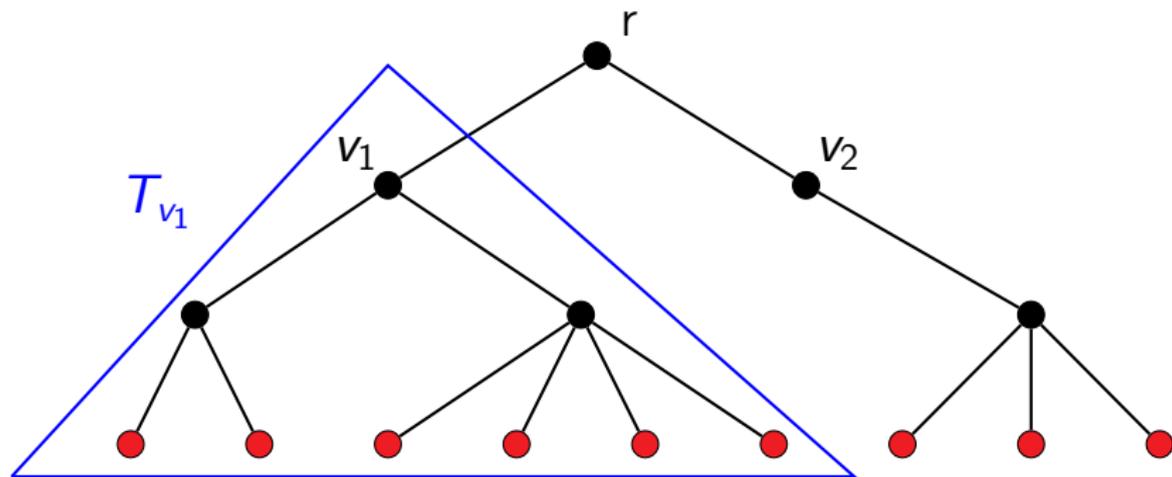


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Crucial question

When playing in T_v for the first time, how many vertices should be probed?

Example: What if T_{v_1} is a big star, while $T_{v_2}, \dots, T_{v_{1000}}$ are much smaller stars?

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Two main parameters for each T_v (assuming target on a leaf):

- 1 $\lambda_k(T_i)$: min. # of steps needed, probing at most k vertex each step;
- 2 $\pi_k(T_i)$: min. # of vertices that can be probed **during the first step** in a strategy taking $\lambda_k(T_i)$ steps.

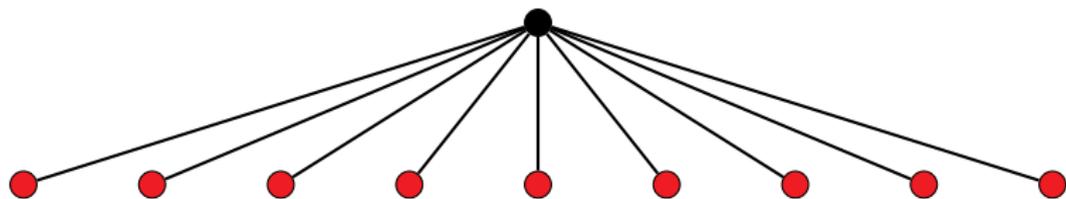
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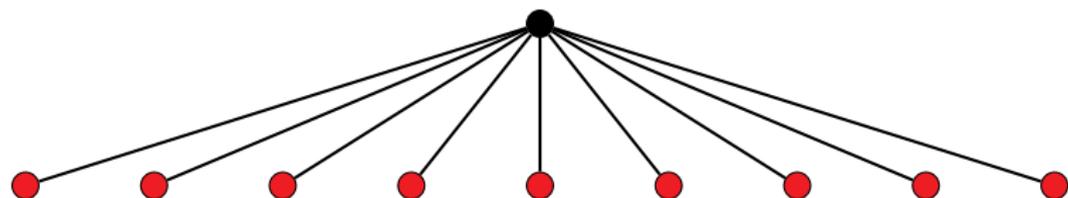


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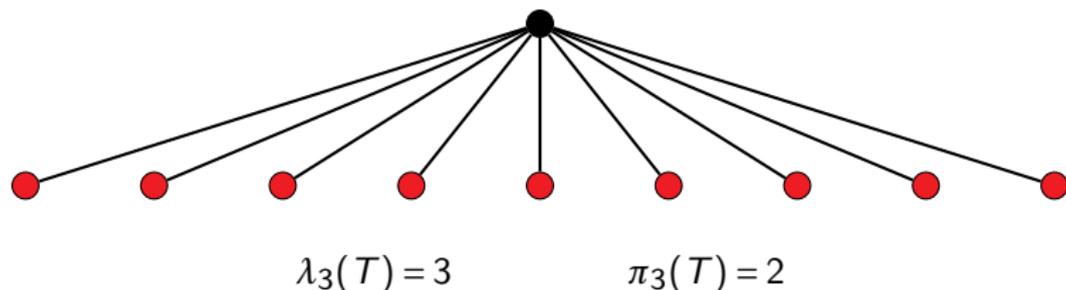
$$\lambda_3(T) = 3$$

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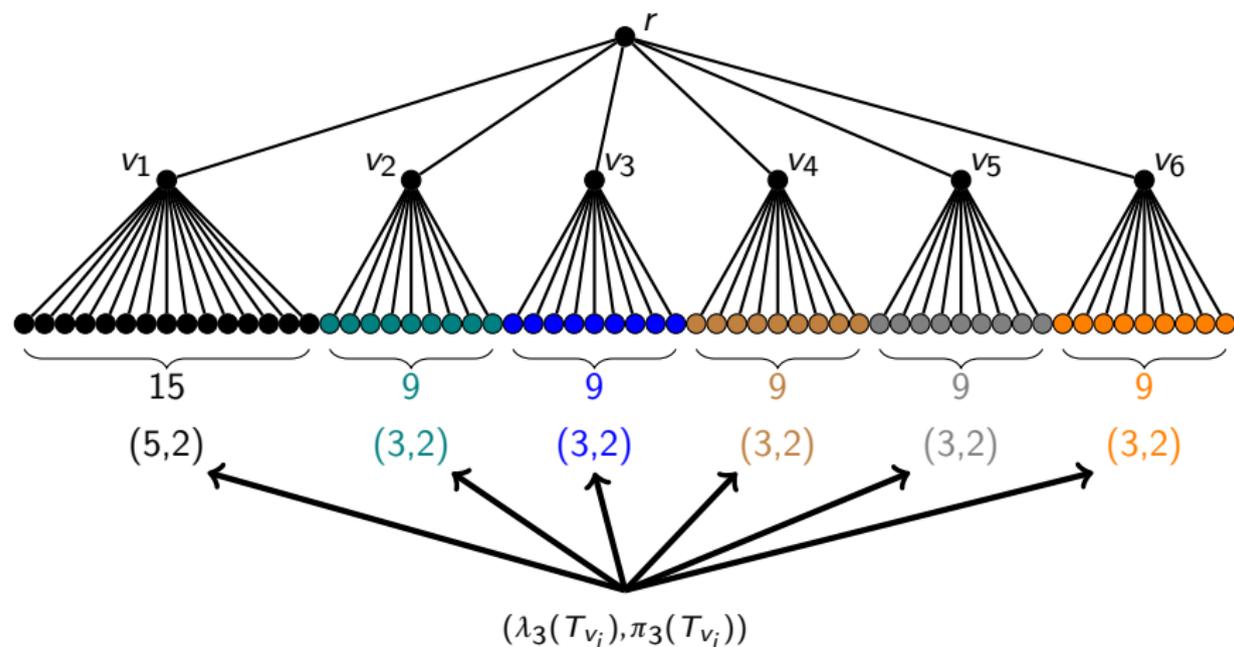
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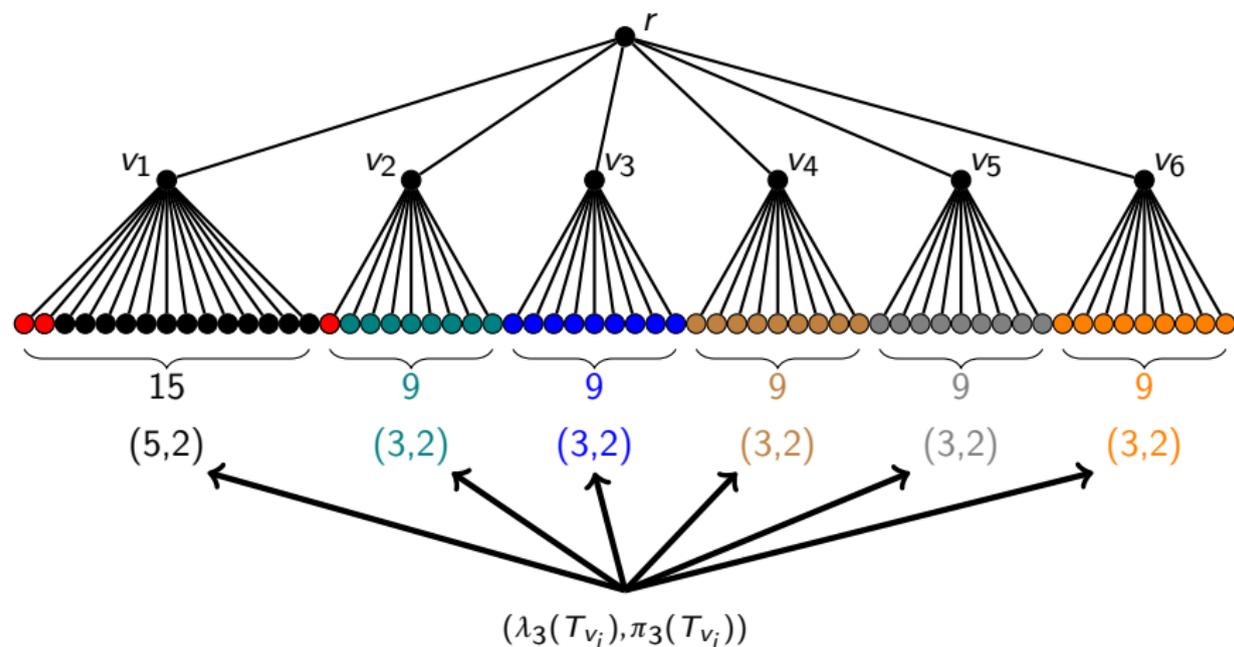
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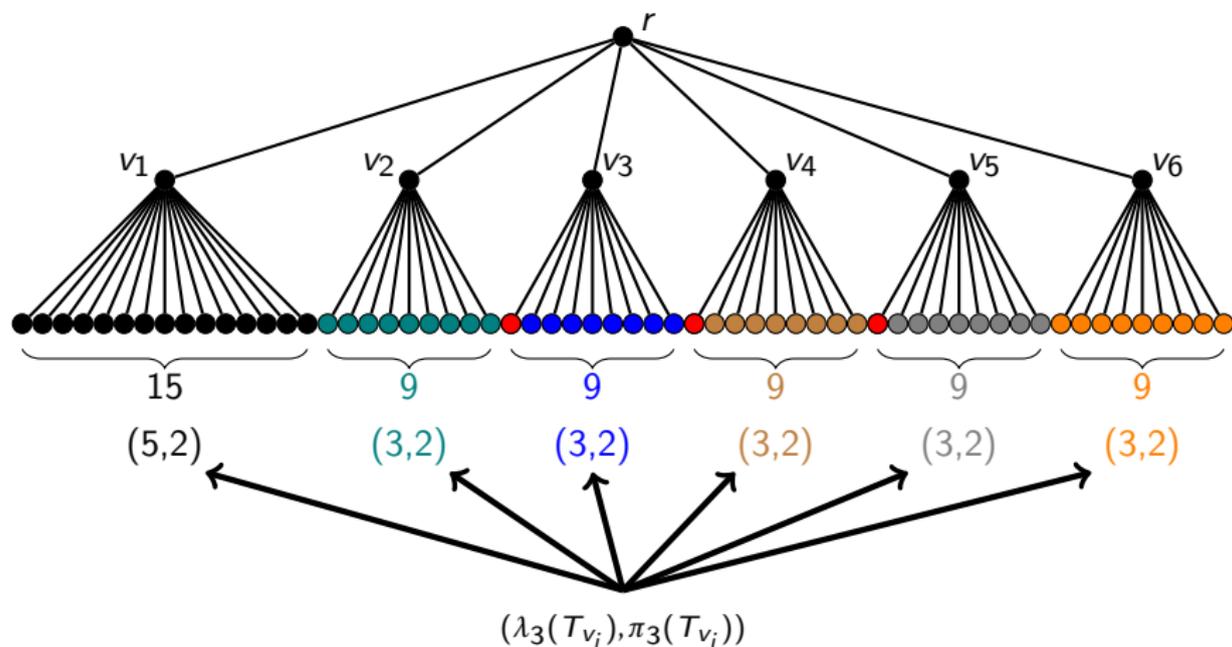
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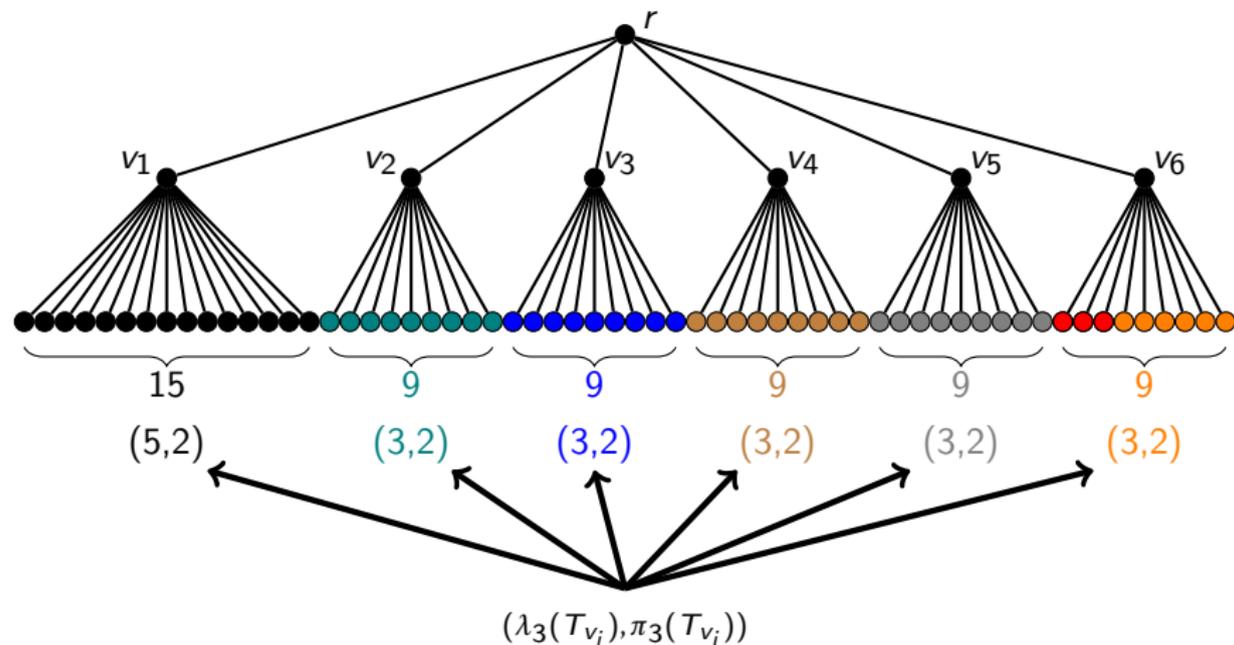
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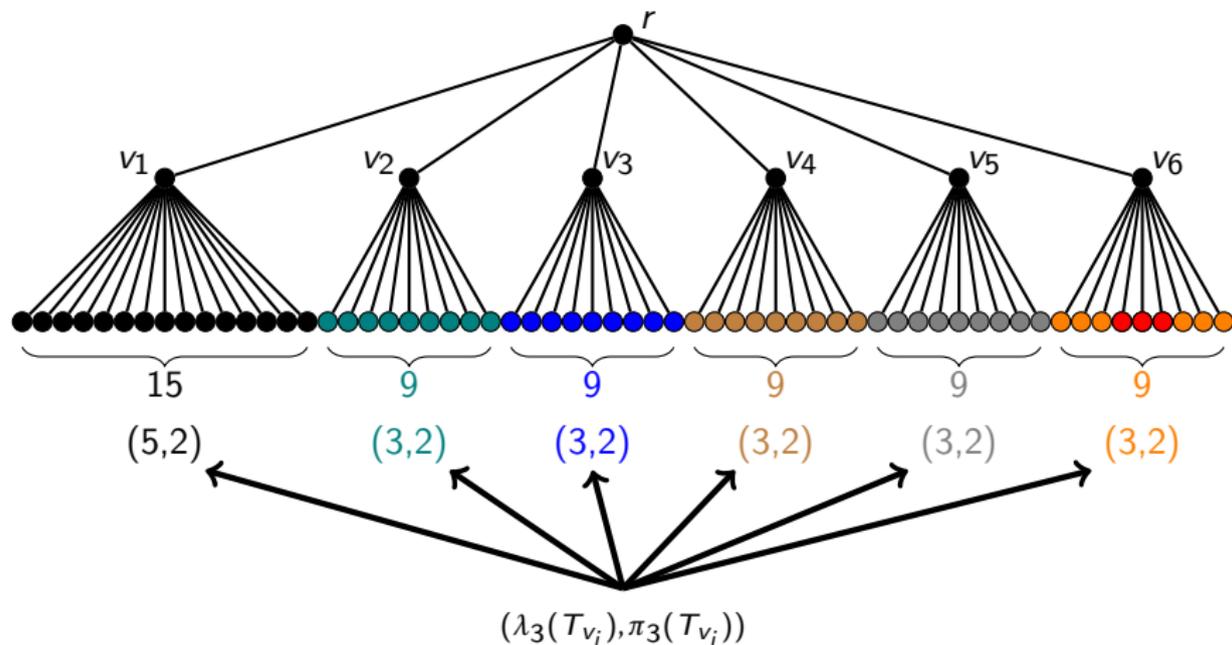
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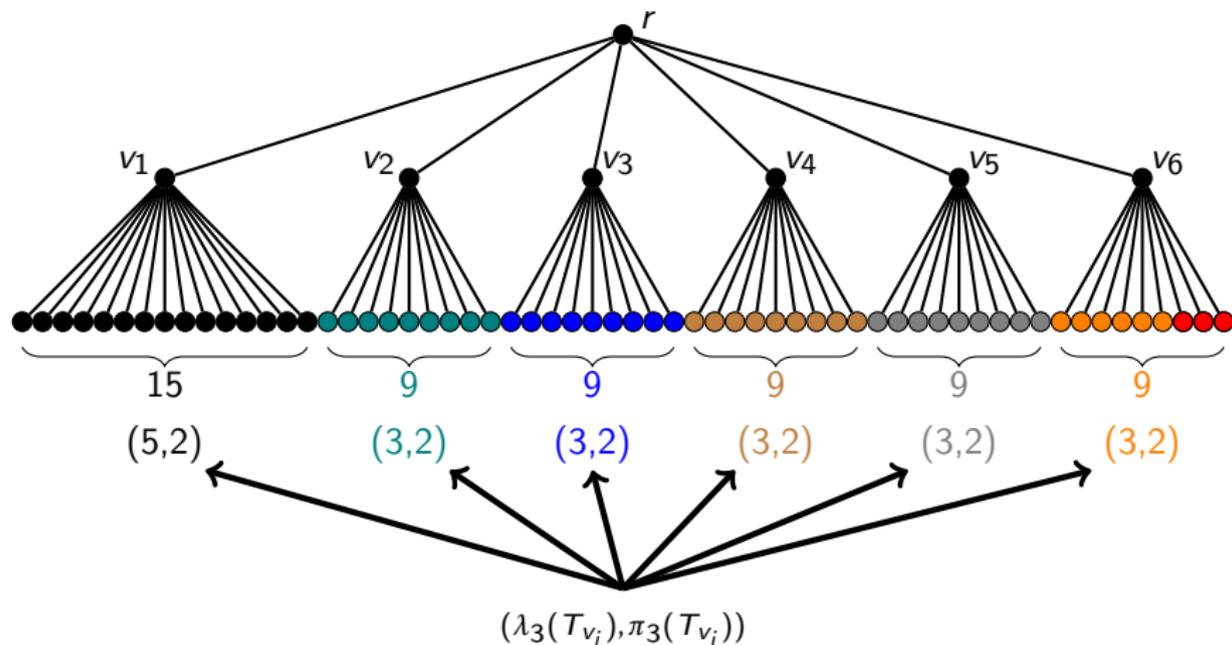
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Also:

- For each pair $(\lambda_k(T_v), \pi_k(T_v))$, can retrieve corresponding strategies.
- $\binom{n}{k}$ possible first steps; polynomial when k is a constant.

Conclusion and perspectives

- Sequential metric dimension of more classes of graphs?

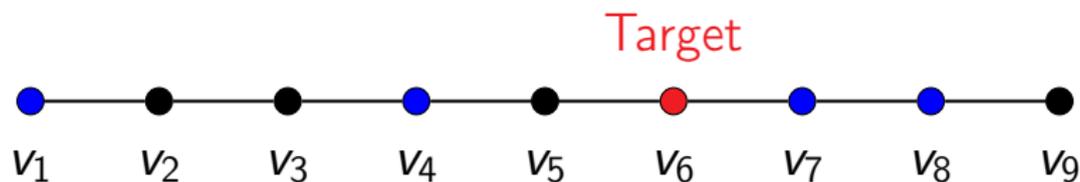
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Thank you for your attention!

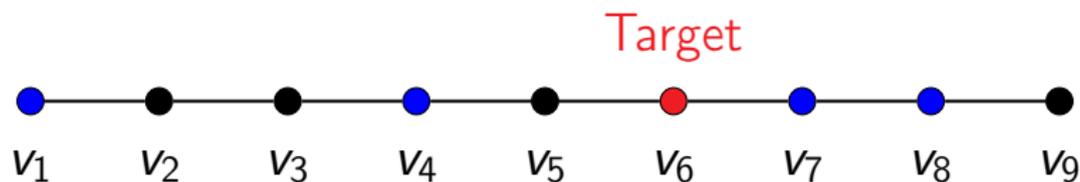
From probed vertices, get **relative distances** to the target instead.



Each $v_i \rightarrow$ **Vector of relative distances** to the probed vertices:

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Arising notions of:

- **Centroidal set** (Foucaud, Klasing, Slater, 2014);
- **Centroidal dimension** (Foucaud, Klasing, Slater, 2014);
- **Sequential centroidal dimension** (us, 2018+).

Decision problems related to the last notion are NP-complete...